

An L_p -theory of stochastic parabolic equations with the random fractional Laplacian driven by Lévy processes

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Abstract

In this paper we give an L_p -theory for stochastic parabolic equations with random fractional Laplacian operator. The driving noises are general Lévy processes.

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1 Introduction

Let $d, m \geq 1$ be positive integers, $p \in [2, \infty)$ and $\alpha \in (0, 2)$. As usual \mathbb{R}^d stands for the Euclidean space of points $x = (x^1, \dots, x^d)$. We will use dx to denote the Lebesgue measure in either \mathbb{R}^d or \mathbb{R}^m , which is clear in each context.

In this article we are dealing with L_p -theory of the stochastic partial differential equations of the type

$$du = \left(a(\omega, t) \Delta^{\alpha/2} u + f(u) \right) dt + \sum_{k=1}^{\infty} g^k(u) \cdot dZ_t^k, \quad u(0) = u_0 \quad (1.1)$$

given for $\omega \in \Omega, t \geq 0$ and $x \in \mathbb{R}^d$. Here Ω is a probability space, $\Delta^{\alpha/2}$ is the fractional Laplacian defined in (2.2), Z_t^k are independent m -dimensional Lévy processes, and the functions f and vector-valued function $g^k = (g^{k,1}, \dots, g^{k,m})$ depend on (ω, t, x, u) satisfying certain continuity conditions.

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Our result will cover the case

$$\begin{aligned} f(u) &= b(\omega, t, x) \Delta^{\beta_1/2} u + c^i(\omega, t, x) u_{x^i} I_{\alpha>1} + d(\omega, t, x) u + f_0, \\ g^{k,j}(u) &= \sigma^{k,j}(\omega, t, x) \Delta^{\beta_2^j/2} u + \nu^{k,j}(\omega, t, x) u + g_0^{k,j}, \quad j = 1, \dots, m. \end{aligned}$$

where $\beta_1 < \alpha$ and $\beta_2^j < \alpha/2$ (see Assumptions 2.13 and 3.7).

An L_p -theory of (1.1) is introduced in [7] for the case that $a(\omega, t) = 1$ and are only finitely many Wiener processes appear in the equation. The approach in [7] cannot cover the case when there are infinitely many Wiener processes, and the assumptions on g in [7] are stronger than conditions in our paper (See Remark 2.12 below). Moreover equations driven by jump processes are not considered in [7]. A Hölder space theory for more general (but non-random) integro-differential equations driven by Hilbert space-valued Wiener process is given in [19] (also see [18] for a deterministic equation). Even though the main result in [19] provide a nice Hölder regularity of the solution to such problem, due to the Hölder-type function spaces defined there, assumptions on f and g are quite strong. Furthermore in [19] the equations with discontinuous Lévy processes are not considered. We emphasize that the approach of this paper, based on L_p theory in [14], is different from [19]. Our results include the case when f and g are only distributions and the number of derivatives of f and g are negative and fractional. On the other hand if f and g are sufficiently smooth in x then Sobolev embedding theorem combined with our L_p -theory gives pointwise Hölder continuity of the solution even when Z^k are general Lévy processes.

L_p -theory for second-order stochastic parabolic equations driven by Wiener processes was first established by Krylov [14]. Recently in [6] L_p regularity theory for second-order stochastic parabolic equations driven by Lévy processes is discussed.

In this paper, we establish an L_p -theory for stochastic parabolic equations with the random fractional Laplacian driven by arbitrary Lévy processes. Our result includes the case when the equation is driven by Lévy space-time white noise (see Theorems 4.3 and 4.4). Among main tools used in the article to study L_p -regularity theory are Burkholder-Davis-Gundy inequality and a parabolic version of Littlewood-Paley inequality for the fractional Laplacian operator introduced [11].

The organization of this article is as follows. First, in section 2, we prove uniqueness and existence results of equation (1.1) driven by Wiener processes in the space $L_p(\Omega \times [0, T], H_p^{\gamma+\alpha/2})$ (Theorem 2.15). Here $p \in [2, \infty)$ and $\gamma \in \mathbb{R}$. In section 3 we extend Theorem 2.15 for the case when Z_t^k are Lévy processes and Z_t^k have finite p -th moments (see condition (3.2)). In section 4, the condition (3.2) is weakened, and the uniqueness and existence results are proved in the space $L_{p,\text{loc}}(\Omega \times [0, T], H_p^{\gamma+\alpha/2})$. The condition (3.2) can be completely dropped if only finitely many Lévy processes appear in the equation.

If we write $c = c(\dots)$, this means that the constant c depends only on what are in parenthesis. The constant c stands for constants whose values are unimportant and which may change from one

appearance to another. The dependence of the lower case constants on the dimensions d, m may not be mentioned explicitly. We will use “:=” to denote a definition, which is read as “is defined to be”. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. Let $C_0^\infty(\mathbb{R}^d)$ be the collection of smooth functions with compact supports in \mathbb{R}^d . Most of functions we discuss in this paper are random (depend on $\omega \in \Omega$). For notational convenience, we suppress the dependency on ω in most of expressions

2 Stochastic Parabolic equations with the random fractional Laplacian driven by Wiener processes

Let (Ω, \mathcal{F}, P) be a complete probability space, $\{\mathcal{F}_t, t \geq 0\}$ be an increasing filtration of σ -fields $\mathcal{F}_t \subset \mathcal{F}$, each of which contains all (\mathcal{F}, P) -null sets. We assume that on Ω we are given independent one-dimensional Wiener processes W_t^1, W_t^2, \dots relative to $\{\mathcal{F}_t, t \geq 0\}$. Let \mathcal{P} be the predictable σ -field generated by $\{\mathcal{F}_t, t \geq 0\}$.

Let $p(t, x)$, where $t > 0$, denote the inverse Fourier transform of $e^{-|\xi|^\alpha t}$ in \mathbb{R}^d , that is,

$$p(t, x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi \cdot x} e^{-|\xi|^\alpha t} d\xi.$$

For a suitable function g and $t > 0$, define the corresponding convolution operator

$$T_t g(x) := (p(t, \cdot) * g(\cdot))(x) := \int_{\mathbb{R}^d} p(t, x - y) g(y) dy, \quad (2.1)$$

and define

$$\partial_x^\alpha g(x) = \Delta^{\frac{\alpha}{2}} g(x) = -(-\Delta)^{\frac{\alpha}{2}} g(x) := \mathcal{F}^{-1}(-|\xi|^\alpha \mathcal{F}(g)(\xi))(x), \quad (2.2)$$

where $\mathcal{F}(g)(\xi) = \hat{g}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} g(x) dx$ is the Fourier transform of g in \mathbb{R}^d .

In this section we study the nonlinear equations of the type

$$du = \left(a(\omega, t) \Delta^{\alpha/2} u + f(u) \right) dt + \sum_{k=1}^{\infty} g^k(u) dW_t^k, \quad u(0) = u_0, \quad (2.3)$$

where $a(\omega, t) \in (\delta, \delta^{-1})$ for some $\delta > 0$, and $f(u) = f(\omega, t, x, u)$ and $g^k(u) = g^k(\omega, t, x, u)$ satisfy certain continuity conditions, which we will put below.

First we introduce some stochastic Banach spaces. Let $(\phi, \psi) := \int_{\mathbb{R}^d} \phi(x) \psi(x) dx$ and for $p \geq 1$,

$$L_p = L_p(\mathbb{R}^d) := \{ \phi : \mathbb{R}^d \rightarrow \mathbb{R}, \|\phi\|_p^p := \int_{\mathbb{R}^d} |\phi(x)|^p dx < \infty \}.$$

For $n = 0, 1, 2, \dots$, define

$$H_p^n = H_p^n(\mathbb{R}^d) := \left\{ u : u, Du, \dots, D^n u \in L_p(\mathbb{R}^d) \right\}.$$

In general, for $\gamma \in \mathbb{R}$ define the space $H_p^\gamma = H_p^\gamma(\mathbb{R}^d) = (1 - \Delta)^{-\gamma/2} L_p$ (called the space of Bessel potentials or the Sobolev space with fractional derivatives) as the set of all distributions u on \mathbb{R}^d such that $(1 - \Delta)^{\gamma/2} u \in L_p$. For $u \in H_p^\gamma$, we define

$$\|u\|_{H_p^\gamma} := \|(1 - \Delta)^{\gamma/2} u\|_p := \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\gamma/2} \mathcal{F}(u)(\xi)]\|_p, \quad (2.4)$$

where \mathcal{F} is the Fourier transform in \mathbb{R}^d . For ℓ_2 -valued $g = (g^1, g^2, \dots)$, we define

$$\|g\|_{H_p^\gamma(\ell_2)} := \|(1 - \Delta)^{\gamma/2} g\|_{\ell_2} \|p\|_p := \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\gamma/2} \mathcal{F}(g)(\xi)]\|_{\ell_2} \|p\|_p.$$

Let $\overline{\mathcal{P}}$ be the completion of \mathcal{P} with respect to $dP \times dt$, and $\mathbb{H}_p^\gamma(T) := L_p(\Omega \times [0, T], \overline{\mathcal{P}}, H_p^\gamma)$, that is, $\mathbb{H}_p^\gamma(T)$ is the set of all $\overline{\mathcal{P}}$ -measurable processes $u : \Omega \times [0, T] \rightarrow H_p^\gamma$ so that

$$\|u\|_{\mathbb{H}_p^\gamma(T)} := \left(\mathbb{E} \left[\int_0^T \|u(\omega, t)\|_{H_p^\gamma}^p dt \right] \right)^{1/p} < \infty.$$

Lemma 2.1 *For any $\beta > 0$, $\eta_\beta^1(\xi) := \frac{(1+|\xi|^2)^{\beta/2}}{1+|\xi|^\beta}$, $\eta_\beta^2 = (\eta_\beta^1)^{-1}$, $\eta^3 := \frac{|\xi|^\beta}{1+|\xi|^\beta}$ and $\eta^4 := \frac{|\xi|^\beta}{(1+|\xi|^2)^{\beta/2}}$ are $L^p(\mathbb{R}^d)$ -multipliers, that is,*

$$\|\mathcal{F}^{-1}(\eta_\beta^i(\xi)(\mathcal{F}u)(\xi))\|_{L_p} \leq c(p, \beta) \|u\|_{L_p}, \quad i = 1, 2, 3, 4.$$

Proof. See Theorem 0.2.6 of [24] (also see the remark below the theorem). \square

Lemma 2.36 easily yields the following results.

Corollary 2.2 (i) *Let $\gamma \geq 0$. There exists a constant $c = c(\gamma) > 0$ so that*

$$c \|u\|_{H_p^\gamma} \leq (\|u\|_{L_p} + \|\partial_x^{\gamma/2} u\|_{L_p}) \leq c^{-1} \|u\|_{H_p^\gamma}.$$

(ii) *For any $\beta \in \mathbb{R}$,*

$$\|\Delta^{\alpha/2} u\|_{H_p^\beta} \leq c(\alpha, \beta) \|u\|_{H_p^{\beta+\alpha}}.$$

Remark 2.3 *Let $\gamma, \beta \geq 0$. Then due to the well-known inequality*

$$\|u\|_{L_p} \leq \varepsilon \|u\|_{H_p^\gamma} + c(\varepsilon, \gamma, \beta) \|u\|_{H_p^{-\beta}},$$

it also follows

$$\|u\|_{H_p^\gamma} \leq c(\gamma, \beta, p) (\|u\|_{H_p^{-\beta}} + \|\partial_x^{\gamma/2} u\|_{L_p}).$$

For ℓ_2 -valued $\overline{\mathcal{P}}$ -measurable processes $g = (g^1, g^2, \dots)$, we write $g \in \mathbb{H}_p^\gamma(T, \ell_2)$ if

$$\|g\|_{\mathbb{H}_p^\gamma(T, \ell_2)} := \left(\mathbb{E} \int_0^T \|(1 - \Delta)^{\gamma/2} g(\omega, t)\|_{\ell_2}^p dt \right)^{1/p} < \infty. \quad (2.5)$$

Denote $\mathbb{L}_p(T) := \mathbb{H}_p^0(T)$ and $\mathbb{L}_p(T, \ell_2) = \mathbb{H}_p^0(T, \ell_2)$. Finally, we say $u_0 \in U_p^\gamma$ if u_0 is \mathcal{F}_0 -measurable function $\Omega \rightarrow H_p^\gamma$ and

$$\|u_0\|_{U_p^\gamma} := \left(\mathbb{E} [\|u_0\|_{H_p^\gamma}^p] \right)^{1/p} < \infty.$$

Remark 2.4 It is easy to check (see Remark 3.2 in [14] for detailed proof) that for any $\gamma \in (-\infty, \infty)$, $g \in \mathbb{H}_p^\gamma(T, \ell_2)$ and $\phi \in C_0^\infty(\mathbb{R}^d)$ we have $\sum_{k=1}^\infty \int_0^T (g^k(\omega, t), \phi)^2 dt < \infty$ a.s., and consequently the series of stochastic integral $\sum_{k=1}^\infty \int_0^t (g^k(\omega, s), \phi) dW_s^k$ converges uniformly in t in probability on $[0, T]$.

Definition 2.5 Write $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$ if $u \in \mathbb{H}_p^{\gamma+\alpha}(T)$, $u(0) \in U_p^{\gamma+\alpha-\alpha/p}$, and for some $f \in \mathbb{H}_p^\gamma(T)$ and $g \in \mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)$

$$du = f dt + \sum_{k=1}^\infty g^k dW_t^k, \quad \text{for } t \in [0, T]$$

in the sense of distributions, that is, for any $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$(u(t), \phi) = (u(0), \phi) + \int_0^t (f(s), \phi) ds + \sum_{k=1}^\infty \int_0^t (g^k(s), \phi) dW_s^k \quad (2.6)$$

holds for all $t \leq T$ a.s.. In this case we write

$$\mathbb{D}u := f, \quad \mathbb{S}^k u := g^k, \quad \mathbb{S}u := (\mathbb{S}^1 u, \dots, \mathbb{S}^k u, \dots)$$

and define the norm

$$\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(T)} := \|u\|_{\mathbb{H}_p^{\gamma+\alpha}(T)} + \|\mathbb{D}u\|_{\mathbb{H}_p^\gamma(T)} + \|\mathbb{S}u\|_{\mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)} + \|u(0)\|_{U_p^{\gamma+\alpha-\alpha/p}}. \quad (2.7)$$

Theorem 2.6 The space $\mathcal{H}_p^{\gamma+\alpha}(T)$ is a Banach space, and for every $0 < t \leq T$

$$\mathbb{E} \left[\sup_{s \leq t} \|u(s, \cdot)\|_{H_p^\gamma}^p \right] \leq c(p, T, \alpha) \left(\|\mathbb{D}u\|_{\mathbb{H}_p^\gamma(t)}^p + \|\mathbb{S}u\|_{\mathbb{H}_p^{\gamma+\alpha/2}(t, \ell_2)}^p + \|u(0)\|_{U_p^\gamma}^p \right). \quad (2.8)$$

In particular, for any $t \leq T$,

$$\|u\|_{\mathbb{H}_p^\gamma(t)}^p \leq c(p, T, \alpha) \int_0^t \|u\|_{\mathcal{H}_p^{\gamma+\alpha}(s)}^p ds. \quad (2.9)$$

Remark 2.7 Note that α is not involved in (2.8).

Proof. See Theorem 3.7 in [14]. Actually in [14] the theorem is proved only for $\alpha = 2$, but the proof works for any $\alpha \in (0, 2)$. We will give the detailed proof of Theorem 3.4 below, which is the counterpart of Theorem 2.6 for pure-jump Lévy processes. \square

Remark 2.8 It follows from (2.4) that for any $\mu, \gamma \in \mathbb{R}$, the operator $(1 - \Delta)^{\mu/2} : H_p^\gamma \rightarrow H_p^{\gamma-\mu}$ is an isometry. Indeed,

$$\|(1 - \Delta)^{\mu/2} u\|_{H_p^{\gamma-\mu}} = \|(1 - \Delta)^{(\gamma-\mu)/2} (1 - \Delta)^{\mu/2} u\|_p = \|(1 - \Delta)^{\gamma/2} u\|_p = \|u\|_{H_p^\gamma}.$$

The same reason shows that $(1 - \Delta)^{\mu/2} : \mathcal{H}_p^\gamma(T) \rightarrow \mathcal{H}_p^{\gamma-\mu}(T)$ is an isometry.

Theorem 2.9 (i) For any deterministic functions $f = f(t, x)$ and $u_0 = u_0(x)$ with

$$\int_0^T \|f(t, \cdot)\|_{H_p^\gamma}^p dt < \infty, \quad \|u_0\|_{H_p^{\gamma+\alpha-\alpha/p}} < \infty,$$

the (deterministic) equation

$$u_t = \Delta^{\alpha/2} u + f, \quad u(0) = u_0$$

has a unique solution u with $\int_0^T \|u(t, \cdot)\|_{H_p^{\gamma+\alpha}}^p dt < \infty$, and for every $0 < t \leq T$

$$\int_0^t \|u(s, \cdot)\|_{H_p^{\gamma+\alpha}}^p ds \leq c(p, T) \left(\int_0^t \|f(s, \cdot)\|_{H_p^\gamma}^p ds + \|u_0\|_{H_p^{\gamma+\alpha-\alpha/p}}^p \right). \quad (2.10)$$

(ii) For any $f \in \mathbb{H}_p^\gamma(T)$ and $u_0 \in U_p^{\gamma+\alpha-\alpha/p}$, the equation

$$u_t = \Delta^{\alpha/2} u + f, \quad u(0) = u_0 \quad (2.11)$$

has a unique solution $u \in \mathbb{H}_p^{\gamma+\alpha}(T)$ and for every $0 < t \leq T$

$$\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(t)} \leq c(p, T) \left(\|f\|_{\mathbb{H}_p^\gamma(t)} + \|u_0\|_{U_p^{\gamma+\alpha-\alpha/p}} \right). \quad (2.12)$$

Proof. (i). See, for instance, Theorem 2.1 in [18].

(ii). This result is also known. See, for instance, Lemma 3.2 and Lemma 3.4 of [7]. Actually since equation (2.11) is deterministic for each fixed ω , the claim of (ii) can be obtained from (i). Indeed, the uniqueness and estimate (2.12) are obvious by (i). For the existence of solution, assume that u_0 and f are sufficiently smooth in x , then using Fourier transform one can easily check that

$$u(t) := T_t u_0 + \int_0^t T_{t-s} f ds$$

solves (2.11) and is in $\mathcal{H}_p^{\gamma+\alpha}(T)$. For general u_0 and f it is enough to use a standard approximation argument (see, for instance, the proof Theorem 2.11). □

Now we give our assumption on $a(\omega, t)$.

Assumption 2.10 The process $a(\omega, t)$ is predictable and there is a constant $\delta > 0$ so that

$$\delta < a(\omega, t) < \delta^{-1}, \quad \forall \omega, t.$$

Now we present an L_p -theory for linear stochastic parabolic equations with random fractional Laplacian.

Theorem 2.11 *Let $p \in [2, \infty)$ and $\gamma \in \mathbb{R}$. For any $f \in \mathbb{H}_p^\gamma(T)$, $g \in \mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)$ and $u_0 \in U_p^{\gamma+\alpha-\alpha/p}$, the linear equation*

$$du = \left(a(\omega, t) \Delta^{\alpha/2} u + f \right) dt + \sum_{k=1}^{\infty} g^k dW_t^k, \quad u(0) = u_0, \quad (2.13)$$

admits a unique solution u in $\mathcal{H}_p^{\gamma+\alpha}(T)$, and for this solution

$$\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(T)} \leq c(p, T, \delta) \left(\|f\|_{\mathbb{H}_p^\gamma(T)} + \|g\|_{\mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)} + \|u_0\|_{U_p^{\gamma+\alpha-\alpha/p}} \right). \quad (2.14)$$

Remark 2.12 (i) *Recall that the unique solution $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$ is understood in the sense of distributions as in Definition 2.5, that is, for any $\phi \in C_0^\infty(\mathbb{R}^d)$,*

$$(u(t), \phi) = (u(0), \phi) + \int_0^t \left(a(\omega, s) (u, \Delta^{\alpha/2} \phi) + (f(s), \phi) \right) ds + \sum_{k=1}^{\infty} \int_0^t (g^k(s), \phi) dW_s^k$$

holds for all $t \leq T$ a.s..

(ii) *A version of Theorem 2.11 is proved in [7] under stronger conditions on g and the processes. Precisely in [7] it is assumed that $a(\omega, t) = 1$, $g \in \mathbb{H}_p^{\gamma+\alpha/2+\varepsilon}(T)$, $\varepsilon > 0$, and there are only finitely many Wiener processes in equation (2.13).*

Proof. Step 1. Owing to Remark 2.8, we only need to show that the theorem holds for a particular $\gamma = \gamma_0$. Indeed, suppose that the theorem holds when $\gamma = \gamma_0$. Then it is enough to notice that $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$ is a solution of the equation if and only if $\bar{u} := (1 - \Delta)^{(\gamma-\gamma_0)/2} u \in \mathcal{H}_p^{\gamma_0+\alpha}$ is a solution of the equation with

$$\bar{f} := (1 - \Delta)^{\frac{(\gamma-\gamma_0)}{2}} f, \quad \bar{g} := (1 - \Delta)^{\frac{(\gamma-\gamma_0)}{2}} g, \quad \bar{u}_0 := (1 - \Delta)^{\frac{(\gamma-\gamma_0)}{2}} u_0,$$

in place of f, g and u_0 , respectively. Furthermore,

$$\begin{aligned} \|u\|_{\mathcal{H}_p^{\gamma+\alpha}(T)} &= \|\bar{u}\|_{\mathcal{H}_p^{\gamma_0+\alpha}(T)} \leq c \left(\|\bar{f}\|_{\mathbb{H}_p^{\gamma_0}(T)} + \|\bar{g}\|_{\mathbb{H}_p^{\gamma_0+\alpha/2+\varepsilon_0}(T, \ell_2)} + \|\bar{u}_0\|_{U_p^{\gamma_0+\alpha-\alpha/p}} \right) \\ &= c \left(\|f\|_{\mathbb{H}_p^\gamma(T)} + \|g\|_{\mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_0}(T, \ell_2)} + \|u_0\|_{U_p^{\gamma+\alpha-\alpha/p}} \right). \end{aligned}$$

Step 2. Next we assume $a(\omega, t) = 1$ and prove the theorem for the equation:

$$du = \Delta^{\alpha/2} u dt + \sum_{k=1}^{\infty} g^k dW_t^k, \quad u(0) = 0. \quad (2.15)$$

Remember that we may assume $\gamma = -\alpha/2$. Since the uniqueness of (2.15) follows from results for the deterministic equations (Theorem 2.9), we only need to show that there exists a solution $u \in \mathbb{H}_p^{\alpha/2}(T)$ of (2.15) and u satisfies estimate (2.14) with $f = u_0 = 0$ and $\gamma = -\alpha/2$.

For a moment, assume $N_0 > 0$ is a fixed non-random constant, $g^k = 0$ for all $k > N_0$ and

$$g^k(t, x) = \sum_{i=0}^{m_k} I_{(\tau_i^k, \tau_{i+1}^k]}(t) g^{k_i}(x) \quad \text{for } k \leq N_0, \quad (2.16)$$

where τ_i^k are bounded stopping times and $g^{k_i}(x) \in C_0^\infty(\mathbb{R}^d)$. Define

$$v(t, x) := \sum_{k=1}^{N_0} \int_0^t g^k(s, x) dW_s^k = \sum_{k=1}^{N_0} \sum_{i=1}^{m_k} g^{k_i}(x) (W_{t \wedge \tau_{i+1}^k}^k - W_{t \wedge \tau_i^k}^k)$$

and

$$u(t, x) := v(t, x) + \int_0^t \Delta^{\alpha/2} T_{t-s} v(s, x) ds = v(t, x) + \int_0^t T_{t-s} \Delta^{\alpha/2} v(s, x) ds. \quad (2.17)$$

Using Fourier transform one can easily show (See, for instance, [7]) that if functions $h_1 = h_1(t, x)$ and $h_2 = h_2(x)$ are sufficiently smooth in x then

$$w_1(t, x) := \int_0^t T_{t-s} h_1(s, x) ds, \quad w_2(t, x) = T_t h_2(x)$$

solve

$$\begin{aligned} dw_1 &= (\Delta^{\alpha/2} w_1 + h_1) dt, \quad w_1(0) = 0, \\ dw_2 &= \Delta^{\alpha/2} w_2 dt, \quad w_2(0) = h_2. \end{aligned}$$

Therefore we have $d(u - v) = (\Delta^{\alpha/2}(u - v) + \Delta^{\alpha/2}v)dt = \Delta^{\alpha/2}u dt$, and

$$du = \Delta^{\alpha/2}u dt + dv = \Delta^{\alpha/2}u dt + \sum_{k=1}^{N_0} g^k dW_t^k.$$

Also by (2.17) and stochastic Fubini theorem ([22, Theorem 64]), almost surely,

$$\begin{aligned} u(t, x) &= v(t, x) + \sum_{k=1}^{N_0} \int_0^t \int_0^s \Delta^{\alpha/2} T_{t-s} g^k(r, x) dW_r^k ds \\ &= v(t, x) - \sum_{k=1}^{N_0} \int_0^t \int_r^t \frac{\partial}{\partial s} T_{t-s} g^k(r, x) ds dW_r^k \\ &= \sum_{k=1}^{N_0} \int_0^t T_{t-s} g^k(s, x) dW_s^k. \end{aligned} \quad (2.18)$$

Hence,

$$\partial_x^{\alpha/2} u(t, x) = \sum_{k=1}^{N_0} \int_0^t \partial_x^{\alpha/2} T_{t-s} g^k(s, \cdot)(x) dW_s^k,$$

and by Burkholder-Davis-Gundy's inequality, we have

$$\mathbb{E} \left[|\partial_x^{\alpha/2} u(t, x)|^p \right] \leq c(p) \mathbb{E} \left[\left(\int_0^t \sum_{k=1}^{N_0} |\partial_x^{\alpha/2} T_{t-s} g^k(s, \cdot)(x)|^2 ds \right)^{p/2} \right].$$

Now we use a parabolic version of Littlewood-Paley inequality for fractional Laplacian (Theorem 2.3 in [11])

$$\int_{\mathbb{R}^d} \int_0^T \left[\int_0^t |\partial_x^{\alpha/2} T_{t-s} g(s, \cdot)(x)|_{\ell_2}^2 ds \right]^{p/2} dt dx \leq c(\alpha, p) \int_{\mathbb{R}^d} \int_0^T |g(t, x)|_{\ell_2}^p dt dx \quad (2.19)$$

and get

$$\mathbb{E} \left[\int_0^T \|\partial_x^{\alpha/2} u(t, \cdot)\|_p^p dt \right] \leq c(p) \mathbb{E} \left[\int_0^T \|g(t, \cdot)\|_{\ell_2}^p dt \right]. \quad (2.20)$$

Similarly, (2.18) and Burkholder-Davis-Gundy's inequality yield

$$\mathbb{E} [|u(t, x)|^p] \leq c(p) \mathbb{E} \left[\left(\int_0^t \sum_{k=1}^{N_0} |T_{t-s} g^k(s, x)|^2 ds \right)^{p/2} \right]. \quad (2.21)$$

Since $(\sum_{k=1}^{N_0} |a_n|^2)^{p/2} \leq c(N_0, p) \sum_{k=1}^{N_0} |a_n|^p$ and $\|T_t f\|_p \leq c\|f\|_p$, we see that for every $t > 0$

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\int_0^t \sum_{k=1}^{N_0} |T_{t-s} g^k(s, x)|^2 ds \right)^{p/2} dx &\leq t^{p/2-1} \int_{\mathbb{R}^d} \int_0^t \left(\sum_{k=1}^{N_0} |T_{t-s} g^k(s, x)|^2 \right)^{p/2} dt dx \\ &\leq c(T, N_0, p) \int_0^t \int_{\mathbb{R}^d} \sum_{k=1}^{N_0} |g^k(t, x)|^p dx dt. \end{aligned}$$

Consequently,

$$\mathbb{E} \int_0^T \int_{\mathbb{R}^d} |u(t, x)|^p dx dt \leq c(T, N_0, p) \mathbb{E} \int_0^T \int_{\mathbb{R}^d} |g(t, x)|_{\ell_2}^p dx ds. \quad (2.22)$$

Thus we proved $\partial_x^{\alpha/2} u, u \in \mathbb{L}_p(T)$, and hence by Corollary 2.2 we have $u \in \mathcal{H}_p^{\alpha/2}(T)$. Note that by Corollary 2.2(ii)

$$\|\Delta^{\alpha/2} u\|_{H_p^{-\alpha/2}} = \|\Delta^{\alpha/4} (\partial_x^{\alpha/2} u)\|_{H_p^{-\alpha/2}} \leq c \|\partial_x^{\alpha/2} u\|_{L_p}.$$

By definition (2.7) and Remark 2.3, for any $t \leq T$,

$$\begin{aligned} \|u\|_{\mathcal{H}_p^{\alpha/2}(t)}^p &\leq c(p) \left(\|u\|_{\mathbb{H}_p^{\alpha/2}(t)}^p + \|\Delta^{\alpha/2} u\|_{\mathbb{H}_p^{-\alpha/2}(t)}^p + \|g\|_{\mathbb{L}_p(t, \ell_2)}^p \right) \\ &\leq c \left(\|u\|_{\mathbb{H}_p^{-\alpha/2}(t)}^p + \|\partial_x^{\alpha/2} u\|_{\mathbb{L}_p(t)}^p + \|g\|_{\mathbb{L}_p(t, \ell_2)}^p \right). \end{aligned} \quad (2.23)$$

Combining this with (2.20) and (2.9) we have that for every $0 < t \leq T$

$$\begin{aligned} \|u\|_{\mathcal{H}_p^{\alpha/2}(t)}^p &\leq c(p, T, \alpha) \left(\|u\|_{\mathbb{H}_p^{-\alpha/2}(t)}^p + \|g\|_{\mathbb{L}_p(t, \ell_2)}^p \right) \\ &\leq c(p, T, \alpha) \int_0^t \|u\|_{\mathcal{H}_p^{\alpha/2}(s)}^p ds + c(p, T, \alpha) \|g\|_{\mathbb{L}_p(T, \ell_2)}^p. \end{aligned} \quad (2.24)$$

Finally, Gronwall leads to (2.14).

Now we drop the additional assumptions on g by using the following standard approximation argument: By Theorem 3.10 in [14], for $g \in \mathbb{L}_p(T, \ell_2)$ we can take a sequence $g_n \in \mathbb{L}_p(T, \ell_2)$ so that $g_n \rightarrow g$ in $\mathbb{L}_p(T, \ell_2)$ and each $g_n = (g_n^1, g_n^2, \dots)$ satisfies above assumed assumptions, that is, $g_n^k = 0$ for all large k and each g_n^k is of type (2.16). By the above result, the equation

$$du_n = \Delta^{\alpha/2} u_n dt + \sum_{k=1}^{\infty} g_n^k dW_t^k, \quad u_n(0) = 0$$

has a unique solution u_n . It also follows that $u_n - u_m$ is the unique solution of

$$d(u_n - u_m) = \Delta^{\alpha/2} (u_n - u_m) dt + (g_n^k - g_m^k) dW_t^k, \quad (u_n - u_m)(0) = 0$$

and, by the previous argument

$$\|u_n - u_m\|_{\mathcal{H}_p^{\alpha/2}(T)} \leq c(p, T) \|g_n - g_m\|_{\mathbb{L}_p(T, \ell_2)}.$$

Consequently, there is $u \in \mathcal{H}_p^{\alpha/2}(T)$ so that $u_n \rightarrow u$ in $\mathcal{H}_p^{\alpha/2}(T)$. We only need to prove u is a solution of (2.15). Equivalently, we need to prove that for any $\phi \in C_0^\infty(\mathbb{R}^d)$, the equality

$$(u(t, \cdot), \phi) = \int_0^t (\Delta^{\alpha/2} u(s, \cdot), \phi) ds + \sum_{k=1}^{\infty} \int_0^t (g^k(s, \cdot), \phi) dW_s^k. \quad (2.25)$$

holds for all $t \leq T$ (a.s.), or equivalently

$$((1 - \Delta)^{-\alpha/2} u(t, \cdot), (1 - \Delta)^{\alpha/2} \phi) = \int_0^t (\Delta^{\alpha/4} u(s, \cdot), \Delta^{\alpha/4} \phi) ds + \sum_{k=1}^{\infty} \int_0^t (g^k(s, \cdot), \phi) dW_s^k. \quad (2.26)$$

By (2.8),

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{t \leq T} \|(1 - \Delta)^{-\alpha/2} (u_n(t, \cdot) - u(t, \cdot))\|_{L_p}^p \right] = 0, \quad \text{a.s.}$$

which implies that one can take a subsequence n_j so that $(1 - \Delta)^{-\alpha/2} u_{n_j} \rightarrow (1 - \Delta)^{-\alpha/2} u$ in $L_p(\mathbb{R}^d)$ uniformly on $[0, T]$ (a.s) and consequently $t \rightarrow ((1 - \Delta)^{-\alpha/2} u(t, \cdot), (1 - \Delta)^{\alpha/2} \phi)$ is continuous on $[0, T]$. By taking the limit from

$$((1 - \Delta)^{-\alpha/2} u_{n_j}(t, \cdot), (1 - \Delta)^{\alpha/2} \phi) = \int_0^t (\Delta^{\alpha/4} u_{n_j}(s, \cdot), \Delta^{\alpha/4} \phi) ds + \sum_{k=1}^{\infty} \int_0^t (g_{n_j}^k(s, \cdot), \phi) dW_s^k$$

and remembering that both sides of (3.7) are continuous in t , one easily get that equality (3.7) holds for all $t \leq T$ (a.s.).

Step 3. Next we prove the theorem for the equation

$$du = (\Delta^{\alpha/2} u + f) dt + g^k dW_t^k, \quad u(0) = u_0. \quad (2.27)$$

Again we may assume $\gamma = -\alpha/2$, and due to Theorem 2.9 we only need to show that there exists a solution u and it satisfies estimate (2.14). By Theorem 2.9, the equation

$$dv = (\Delta^{\alpha/2}v + f)dt, \quad v(0) = u_0$$

has a solution $v \in \mathcal{H}_p^{\alpha/2}(T)$ and

$$\|v\|_{\mathcal{H}_p^{\alpha/2}(T)} \leq c(p, T) \left(\|f\|_{\mathbb{H}_p^{-\alpha/2}(T)} + \|u_0\|_{U_p^{\alpha/2-\alpha/p}} \right).$$

Also by the result of **Step 2**, the equation

$$dw = \Delta^{\alpha/2}w dt + \sum_{k=1}^{\infty} g^k dW_t^k, \quad w(0) = 0$$

has a unique solution and

$$\|w\|_{\mathcal{H}_p^{\alpha/2}(T)} \leq c(p, T) \|g\|_{\mathbb{L}_p(T, \ell_2)}.$$

Now it is enough to take $u = v + w$.

Step 4 (A priori estimate). We prove the a priori estimate (2.14) holds given that a solution $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$ of the following equation already exists :

$$du = \left(a(\omega, t) \Delta^{\alpha/2}u + f \right) dt + \sum_{k=1}^{\infty} g^k dW_t^k, \quad u(0) = u_0.$$

This time we prove (2.14) only for $\gamma = 0$. This is enough due to the reason given in Step 1. By Step 3, the equation

$$dv = (\Delta^{\alpha/2}v + f)dt + \sum_{k=1}^{\infty} g^k dW_t^k, \quad v(0) = u_0.$$

has a solution $v \in \mathcal{H}_p^{\alpha}(T)$ and

$$\|v\|_{\mathcal{H}_p^{\alpha}(T)} \leq c(p, T) \left(\|f\|_{\mathbb{L}_p(T)} + \|g\|_{H_p^{\alpha/2}(T, \ell_2)} + \|u_0\|_{U_p^{\alpha-\alpha/p}} \right).$$

Note that $\bar{u} := u - v$ satisfies

$$d\bar{u} = (a(\omega, t) \Delta^{\alpha/2}\bar{u} + \bar{f})dt, \quad \bar{u}(0) = 0,$$

where $\bar{f} := (a(\omega, t) - 1) \Delta^{\alpha/2}v$, and

$$\|\bar{f}\|_{\mathbb{L}_p(T)} \leq c \|\Delta^{\alpha/2}v\|_{\mathbb{L}_p(T)} \leq c \left(\|f\|_{\mathbb{L}_p(T)} + \|g\|_{H_p^{\alpha/2}(T, \ell_2)} + \|u_0\|_{U_p^{\alpha-\alpha/p}} \right).$$

Since $u = v + \bar{u}$, $\|u\|_{\mathcal{H}_p^{\alpha}(T)} \leq \|v\|_{\mathcal{H}_p^{\alpha}(T)} + \|\bar{u}\|_{\mathcal{H}_p^{\alpha}(T)}$ and $\|\bar{u}\|_{\mathcal{H}_p^{\alpha}(T)} \leq c \|\bar{u}\|_{\mathbb{H}_p^{\alpha}(T)} + c \|\bar{f}\|_{\mathbb{L}_p(T)}$, to prove (2.14) we only need to show that for each $\omega \in \Omega$,

$$\int_0^T \|\bar{u}(t, \cdot)\|_{H_p^{\alpha}}^p dt \leq c(p, T, \alpha, \delta) \int_0^T \|\bar{f}(t, \cdot)\|_{L_p}^p dt. \quad (2.28)$$

For fixed ω , define a non-random functions

$$\tilde{u}(t, x) = \bar{u}(\omega, \xi(\omega, t), x) \quad \tilde{f}(t, x) = a(\omega, t)^{-1} \bar{f}(\omega, \xi(\omega, t), x). \quad (2.29)$$

where $\xi(\omega, t) := \int_0^t \frac{ds}{a(\omega, s)}$. Then clearly \tilde{u} satisfies

$$\tilde{u}_t = \Delta^{\alpha/2} \tilde{u} + \tilde{f}, \quad \tilde{u}(0) = 0.$$

Let $\tilde{T}(\omega, T)$ be such that $T = \int_0^{\tilde{T}(\omega, T)} \frac{ds}{a(\omega, s)}$. Since $\delta T < \tilde{T}(\omega, T) < \delta T$, applying (2.10), we get

$$\int_0^{\tilde{T}(\omega, T)} \|\tilde{u}(t, \cdot)\|_{H_p^\alpha}^p dt \leq c(p, T, \delta, \alpha) \int_0^{\tilde{T}(\omega, T)} \|\tilde{f}(t, \cdot)\|_{L_p}^p dt.$$

This and relations in (2.29) easily lead to (2.28).

Step 5 (Method of continuity). The solvability of equation (2.27), the a priori estimate (2.14) and the method of continuity obviously finish the proof of the theorem. But below we show how the method of continuity works only for reader's convenience.

For $\lambda \in [0, 1]$, denote $a_\lambda(\omega, t) = (1 - \lambda) + \lambda a(\omega, t)$. Then obviously a_λ is predictable and $a_\lambda \in (\delta, \delta^{-1})$. It follows from Step 4 that if $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$ is a solution of the equation

$$du = \left(a_\lambda(\omega, t) \Delta^{\alpha/2} u + f \right) dt + \sum_{k=1}^{\infty} g^k dW_t^k, \quad u(0) = u_0, \quad (2.30)$$

then the estimate (2.14) holds with the same constant $c = c(p, T, \delta)$. Now let J be the collection of $\lambda \in [0, 1]$ so that for any $f \in \mathbb{H}_p^\gamma(T)$, $g \in \mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)$ and $u_0 \in U_p^{\gamma+\alpha/2-\alpha/p}$, equation (2.30) has a solution. By Step 3, $0 \in J$. Note that to finish the proof of the theorem we only need to show $1 \in J$. Now let $\lambda_0 \in J$. Obviously $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$ is a solution of (2.30) if and only if

$$du = \left(a_{\lambda_0}(\omega, t) \Delta^{\alpha/2} u + [(a_\lambda(\omega, t) - a_{\lambda_0}(\omega, t)) \Delta^{\alpha/2} u + f] \right) dt + \sum_{k=1}^{\infty} g^k dW_t^k, \quad u(0) = u_0. \quad (2.31)$$

Now fix $u^1 \in \mathcal{H}_p^{\gamma+\alpha}(T)$ with initial date u_0 (for instance take the solution of (2.27)), and define u^2, u^3, \dots so that $u^{n+1} \in \mathcal{H}_p^{\gamma+\alpha}(T)$ is the solution of

$$du^{n+1} = \left(a_{\lambda_0}(\omega, t) \Delta^{\alpha/2} u^{n+1} + [(a_\lambda(\omega, t) - a_{\lambda_0}(\omega, t)) \Delta^{\alpha/2} u^n + f] \right) dt + g^k dW_t^k, \quad u(0) = u_0. \quad (2.32)$$

Then $v^{n+1} := u^{n+1} - u^n$ satisfies

$$dv^{n+1} = \left(a_{\lambda_0}(\omega, t) \Delta^{\alpha/2} v^{n+1} + (a_\lambda(\omega, t) - a_{\lambda_0}(\omega, t)) \Delta^{\alpha/2} v^n \right) dt$$

By the a priori estimate (2.14),

$$\begin{aligned} \|v^{n+1}\|_{\mathcal{H}_p^{\gamma+\alpha}(T)} &\leq c \|(a_\lambda - a_{\lambda_0}) \Delta^{\alpha/2} v^n\|_{\mathbb{H}_p^\gamma(T)} \\ &\leq N(p, T, \delta) |\lambda - \lambda_0| \|v^n\|_{\mathcal{H}_p^{\gamma+\alpha}(T)}. \end{aligned}$$

Thus if $|\lambda - \lambda_0| < 1/(2N(p, T, \delta))$, the map which send u^n to u^{n+1} is a contraction in $\mathcal{H}_p^{\gamma+\alpha}(T)$, and has a unique fixed point u . Thus u satisfies (2.30)–(2.31). Since the above constant N is independent of λ , it follows that $J = [0, 1]$ and the theorem is proved. \square

Finally we consider the nonlinear equation

$$du = \left(a(\omega, t) \Delta^{\alpha/2} u + f(u) \right) dt + \sum_{k=1}^{\infty} g^k(u) dW_t^k, \quad u(0) = u_0, \quad (2.33)$$

where $f(u) = f(\omega, t, x, u)$ and $g^k(u) = g^k(\omega, t, x, u)$.

Assumption 2.13 Assume $f(0) \in \mathbb{H}_p^\gamma(T)$ and $g(0) \in \mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)$. Moreover, for any $\varepsilon > 0$, there exists a constant K_ε so that for any $u = u(x), v = v(x) \in H_p^{\gamma+\alpha}$ and ω, t , we have

$$\begin{aligned} & \|f(t, \cdot, u(\cdot)) - f(t, \cdot, v(\cdot))\|_{H_p^\gamma} + \|g(t, \cdot, u(\cdot)) - g(t, \cdot, v(\cdot))\|_{H_p^{\gamma+\alpha/2}(\ell_2)} \\ & \leq \varepsilon \|u - v\|_{H_p^{\gamma+\alpha}} + K(\varepsilon) \|u - v\|_{H_p^\gamma}. \end{aligned} \quad (2.34)$$

To give an example of $f(u)$ and $g(u)$ satisfying Assumption 2.13, we introduce the space of point-wise multipliers in H_p^γ . For each $r \geq 0$, define

$$B^r = \begin{cases} B(\mathbb{R}^d) & \text{if } r = 0, \\ C^{r-1,1}(\mathbb{R}^d) & \text{if } r = 1, 2, \dots, \\ C^r(\mathbb{R}^d) & \text{otherwise,} \end{cases} \quad (2.35)$$

where $B(\mathbb{R}^d)$ is the space of bounded Borel measurable functions on \mathbb{R}^d , $C^{r-1,1}(\mathbb{R}^d)$ is the space of $r - 1$ times continuously differentiable functions whose $(r - 1)$ st order derivatives are Lipschitz continuous, and $C^r(\mathbb{R}^d)$ is the usual Hölder space. Also we use the space B^r for ℓ_2 -valued functions. For instance, if $g = (g^1, g^2, \dots)$, then $|g|_{B^0} = \sup_x |g(x)|_{\ell_2}$ and

$$|g|_{C^{n-1,1}} = \sum_{|\alpha| \leq n-1} |D^\alpha g|_{B^0} + \sum_{|\alpha|=n-1} \sup_{x \neq y} \frac{|D^\alpha g(x) - D^\alpha g(y)|_{\ell_2}}{|x - y|}.$$

Fix $\kappa_0 = \kappa_0(\gamma) \geq 0$ so that $\kappa_0 > 0$ if γ is not integer. It is known (see, for instance, Lemma 5.2 in [14]) that for any $a \in B^{|\gamma|+\kappa_0}$ and $h \in H_p^\gamma$,

$$\|ah\|_{H_p^\gamma} \leq c(\gamma, \kappa_0) |a|_{B^{|\gamma|+\kappa_0}} |h|_{H_p^\gamma} \quad (2.36)$$

and the same inequality holds for ℓ_2 -valued functions a .

Example 2.14 Fix $\kappa_0 = \kappa_0(\gamma) \geq 0$ so that $\kappa_0 > 0$ if γ is not integer. Consider

$$f(u) = b(\omega, t, x) \Delta^{\beta_1/2} u + \sum_{i=1}^d c^i(\omega, t, x) u_{x^i} I_{\alpha>1} + d(\omega, t, x) u + f_0,$$

$$g^k(u) = \sigma^k(\omega, t, x) \Delta^{\beta_2/2} u + v^k(\omega, t, x) u + g_0^k,$$

where $\beta_1 < \alpha$, $\beta_2 < \alpha/2$, $f_0 \in \mathbb{H}_p^\gamma(T)$ and $g_0 \in \mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)$. Assume for each ω, t ,

$$|b|_{B^{|\gamma|+\kappa_0}} + \sum_{i=1}^d |c^i|_{B^{|\gamma|+\kappa_0}} + |d|_{B^{|\gamma|+\kappa_0}} + |\sigma|_{B^{|\gamma|+\alpha/2+\kappa_0}} + |\nu|_{B^{|\gamma|+\alpha/2+\kappa_0}} \leq K.$$

Then by (2.36), for each t

$$\begin{aligned} & \|f(t, \cdot, u(\cdot)) - f(t, \cdot, v(\cdot))\|_{H_p^\gamma} + \|g(t, \cdot, u(\cdot)) - g(t, \cdot, v(\cdot))\|_{H_p^{\gamma+\alpha/2}(\ell_2)} \\ & \leq c \left(\|\Delta^{\beta_1/2}(u-v)\|_{H_p^\gamma} + I_{\alpha>1} \|D(u-v)\|_{H_p^\gamma} + \|u-v\|_{H_p^{\gamma+\alpha/2}} + \|\Delta^{\beta_2/2}(u-v)\|_{H_p^{\gamma+\alpha/2}} \right). \end{aligned}$$

Since for any $\alpha_1 < \alpha$ and $\varepsilon > 0$, by interpolation theory,

$$\|u\|_{H_p^{\gamma+\alpha_1}} \leq c(\alpha, \alpha_1) \|u\|_{H_p^{\gamma+\alpha}}^{\alpha_1/\alpha} \|u\|_{H_p^\gamma}^{1-\alpha_1/\alpha} \leq \varepsilon \|u\|_{H_p^{\gamma+\alpha}} + c(\varepsilon, \alpha_1, \alpha) \|u\|_{H_p^\gamma},$$

one easily gets (2.34).

Here is the main result of this section.

Theorem 2.15 *Suppose Assumptions 2.10 and 2.13 hold. Then equation (2.33) has a unique solution $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$, and for this solution*

$$\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(T)} \leq c \left(\|f(0)\|_{\mathbb{H}_p^\gamma(T)} + \|g(0)\|_{\mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)} + \|u_0\|_{U_p^{\gamma+\alpha/2-\alpha/p}} \right),$$

where $c = c(p, T, \delta)$.

Proof. Our proof is virtually identical to the that of Theorem 6.4 in [14], where the theorem is proved when $\alpha = 2$. The only difference is that one has to use Theorem 2.11 in this article, in place the corresponding result in [14]. We skip the proof here since we will give the proof for more general case in next section. \square

By Sobolev embedding theorem, we immediately get the following

Corollary 2.16 *Suppose Assumptions 2.10 and 2.13 hold. If $\gamma + \alpha > d/p$, then the unique solution $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$ of equation (2.33) is $C^{\gamma+\alpha-d/p}$ -valued process on $[0, T] \times \Omega$ a.s..*

3 General case

Let Z_t^1, Z_t^2, \dots be independent m -dimensional Lévy processes relative to $\{\mathcal{F}_t, t \geq 0\}$. For $t \geq 0$ and Borel set $A \in \mathcal{B}(\mathbb{R}^m \setminus \{0\})$, define

$$N_k(t, A) := \# \left\{ 0 \leq s \leq t; Z_s^k - Z_{s-}^k \in A \right\}, \quad \tilde{N}_k(t, A) := N_k(t, A) - t\nu_k(A)$$

where $\nu_k(A) := \mathbb{E}[N_k(1, A)]$ is the Lévy measure of Z^k . By Lévy-Itô decomposition, there exist a vector α^k , a non-negative definite matrix β^k and m -dimensional Wiener process B^k so that

$$Z^k(t) = \alpha^k t + \beta^k B_t^k + \int_{|z|<1} z \tilde{N}_k(t, dz) + \int_{|z|\geq 1} z N_k(t, dz). \quad (3.1)$$

For any $q, k = 1, 2, \dots$, denote

$$\hat{c}_{k,q} := \left(\int_{\mathbb{R}^m} |z|^q \nu_k(dz) \right)^{1/q}.$$

Now we fix $p \in [2, \infty)$ and denote $\hat{c}_k := (\hat{c}_{k,2} \vee \hat{c}_{k,p})$. In this section we assume

$$\hat{c} := \sup_{k \geq 1} \hat{c}_k < \infty. \quad (3.2)$$

((3.2) will be weakened in section 4). Then for any $2 < q < p$, by Hölder's inequality,

$$\hat{c}_{k,q} \leq \left(\int_{\mathbb{R}^m} |z|^2 \nu_k(dz) \right)^{(p-q)/(q(p-2))} \left(\int_{\mathbb{R}^m} |z|^p \nu_k(dz) \right)^{(q-2)/(q(p-2))} \leq \hat{c}_k.$$

By (3.2), $\int_{|z|\geq 1} |z| \nu_k(dz) \leq \int_{|z|\geq 1} |z|^2 \nu_k(dz) < \infty$, and

$$\int_{|z|\geq 1} z N_k(t, dz) = \int_{|z|\geq 1} z \tilde{N}_k(t, dz) + t \int_{|z|\geq 1} z \nu_k(dz).$$

Thus by absorbing $\tilde{\alpha}_k := \int_{|z|\geq 1} z \nu_k(dz)$ into α_k we can rewrite (3.1) as

$$Z_t^k = \tilde{\alpha}_k t + \beta_k B_t^k + \int_{\mathbb{R}^m} z \tilde{N}_k(t, dz).$$

We first consider the following linear equation:

$$du = \left(a(\omega, t) \Delta^{\alpha/2} u + f \right) dt + \sum_{k=1}^{\infty} g^k \cdot dZ_t^k, \quad u(0) = u_0. \quad (3.3)$$

Relocation of the term $\sum_{k=1}^{\infty} g^k \cdot \tilde{\alpha}_k dt$ into the deterministic part of (3.3) allow us to assume $\tilde{\alpha}_k = (0, \dots, 0)$. Moreover, since $B^{k,j}$'s are independent 1-dimensional Wiener processes where $B^k = (B^{k,1}, \dots, B^{k,m})$, (3.3) can be written as

$$du = \left(a(\omega, t) \Delta^{\alpha/2} u + f \right) dt + \sum_{i=1}^{\infty} h^i dW_t^i + \sum_{k=1}^{\infty} \sum_{j=1}^m g^{k,j} dY_t^{k,j}, \quad u(0) = u_0, \quad (3.4)$$

for some $h = (h^1, h^2, \dots)$ and independent one-dimensional Wiener processes W_t^k and $Y_t^k := \int_{\mathbb{R}^m} z \tilde{N}_k(t, dz)$. Note that Y_t^k are independent m -dimensional pure jump Lévy processes with Lévy measure of ν^k .

Furthermore by considering $u - v$, where v is the solution of

$$dv = a(\omega, t) \Delta^{\alpha/2} v dt + \sum_{i=k}^{\infty} h^i dW_t^i, \quad v(0) = 0$$

from Theorem 2.11, we find that without loss of generality we may also assume h^k 's are all zero.

By $\langle M, N \rangle$ we denote the bracket of real-valued square integrable martingales M and N . Also let $[M]$ denote the quadratic variation of M .

Remark 3.1 (i) Note that, if $\hat{c}_{k,2} < \infty$, then $Y^{k,i} = \int_{\mathbb{R}^m} z^i \tilde{N}_k(t, dz)$ is a square integrable martingale for each $k \geq 1$ and $i = 1, \dots, m$. Also for any $\bar{\mathcal{P}}$ -measurable process $H = (H^1, \dots, H^m) \in L_2(\Omega \times [0, T], \mathbb{R}^m)$ which has a predictable version $\bar{H} = (\bar{H}^1, \dots, \bar{H}^m)$,

$$M_t^k := \int_0^t H_s \cdot dY_s^k = \sum_{i=1}^m \int_0^t \int_{\mathbb{R}^m} H_s^i z^i \tilde{N}_k(ds, dz) = \sum_{i=1}^m \int_0^t \int_{\mathbb{R}^m} \bar{H}_s^i z^i \tilde{N}_k(ds, dz)$$

is a square integrable martingale with

$$[M^k]_t = \sum_{i,j=1}^m \int_0^t \int_{\mathbb{R}^m} H^i H^j z^i z^j N(ds, dz),$$

$$E[M^k]_t = \sum_{i,j} \left(\int_{\mathbb{R}^m} z^i z^j \nu^k(dz) \right) \mathbb{E} \int_0^t H^i(s) H^j(s) ds \leq \hat{c}^2 m^2 \mathbb{E} \int_0^t |H_s|^2 ds.$$

(ii) Suppose that (3.2) holds. Then, for any $1 \leq j \leq m$, $g^{j,\cdot} \in \mathbb{H}_p^\gamma(T, \ell_2)$ and $\phi \in C_0^\infty(\mathbb{R}^d)$, the series of stochastic integral

$$\sum_{k=1}^\infty \sum_{j=1}^m \int_0^t (g^{k,j}(s, \cdot), \phi) dY_s^{k,j}$$

defines a square integrable martingale on $[0, T]$, which is right continuous with left limits. Indeed, denote $M_n := \sum_{k=1}^n \sum_{j=1}^m \int_0^t (g^{k,j}(s, \cdot), \phi) dY_s^{k,j}$, then the quadratic variation of M_n is

$$[M_n]_t = \sum_{k=1}^n \sum_{i,j=1}^m \int_0^t \int_{\mathbb{R}^m} (g^{k,i}, \phi) (g^{k,j}(s, \cdot), \phi) z^i z^j N^k(ds, dz),$$

and

$$\mathbb{E}[M_n]_t = \sum_{k=1}^n \sum_{i,j=1}^m \mathbb{E} \int_0^t (g^{k,i}, \phi) (g^{k,j}(s, \cdot), \phi) \int_{\mathbb{R}^m} z^i z^j \nu_k(dz) ds \leq c(m, \hat{c}) \sum_{k=1}^n \sum_{j=1}^m \mathbb{E} \int_0^t (g^{k,j}(s, \cdot), \phi)^2 ds.$$

Also, with $q := p/(p-2)$, for every $1 \leq j \leq m$,

$$\begin{aligned} \sum_{k=1}^\infty \mathbb{E} \left[\int_0^T (g^{k,j}(s, \cdot), \phi)^2 ds \right] &= \sum_{k=1}^\infty \mathbb{E} \left[\int_0^T ((1-\Delta)^{\gamma/2} g^{k,j}(s, \cdot), (1-\Delta)^{-\gamma/2} \phi)^2 ds \right] \\ &\leq \| (1-\Delta)^{-\gamma/2} \phi \|_1 \mathbb{E} \left[\int_0^T \left(\sum_{k=1}^\infty |(1-\Delta)^{\gamma/2} g^{k,j}(s, \cdot)|^2, |(1-\Delta)^{-\gamma/2} \phi| \right) ds \right] \\ &\leq \| (1-\Delta)^{-\gamma/2} \phi \|_1 \| (1-\Delta)^{-\gamma/2} \phi \|_q \mathbb{E} \left[\int_0^T \left\| \sum_{k=1}^\infty |(1-\Delta)^{\gamma/2} g^{k,j}(s, \cdot)|^2 \right\|_{p/2} ds \right] \\ &\leq \| (1-\Delta)^{-\gamma/2} \phi \|_1 \| (1-\Delta)^{-\gamma/2} \phi \|_q T^{1-\frac{2}{p}} \| g^{j,\cdot} \|_{\mathbb{H}_p^\gamma(T, \ell_2)}^2 < \infty. \end{aligned}$$

It follows that $[M_n]_t$ converges in probability uniformly on $[0, T]$ and this certainly proves the claim.

Definition 3.2 Write $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$ if $u \in \mathbb{H}_p^{\gamma+\alpha}(T)$, $u(0) \in U_p^{\gamma+\alpha-\alpha/p}$, and for some $f \in \mathbb{H}_p^\gamma(T)$ $h \in \mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)$ and $g^{k,j} \in \mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)$, $1 \leq j \leq m$

$$du = f dt + \sum_{k=1}^{\infty} h^k dW_t^k + \sum_{k=1}^{\infty} \sum_{j=1}^m g^{k,j} dY_t^{k,j}, \quad u(0) = u_0, \quad \text{for } t \in [0, T]$$

in the sense of distributions, that is, for any $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$(u(t, \cdot), \phi) = (u(0, \cdot), \phi) + \int_0^t (f(s, \cdot), \phi) ds + \sum_{k=1}^{\infty} \int_0^t (h^k(s, \cdot), \phi) ds + \sum_{k=1}^{\infty} \sum_{j=1}^m \int_0^t (g^{k,j}(s, \cdot), \phi) dY_s^{k,j} \quad (3.5)$$

holds for all $t \leq T$ a.s.. In this case we write

$$\mathbb{D}u := f, \quad \mathbb{S}_c u := (h^1, \dots, h^k, \dots), \quad \mathbb{S}_d^{k,j} u := g^{k,j}, \quad \mathbb{S}_d^{j,j} u := (g^{1,j}, \dots, g^{k,j}, \dots)$$

and define

$$\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(T)} := \|u\|_{\mathbb{H}_p^{\gamma+\alpha}(T)} + \|\mathbb{D}u\|_{\mathbb{H}_p^\gamma(T)} + \|\mathbb{S}_c u\|_{\mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)} + \sum_{j=1}^m \|\mathbb{S}_d^{j,j} u\|_{\mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)} + \|u(0)\|_{U_p^{\gamma+\alpha-\alpha/p}}.$$

To prove that $\mathcal{H}_p^{\gamma+\alpha}(T)$ is a Banach space we need the following result, which is an infinite dimensional extension of Kunita's inequality (for example, see [2, Theorem 4.4.23]). In fact, if $m = 1$ then the proof is given in [5].

Lemma 3.3 Suppose $1 \leq j \leq m$, $g^{j,j}(\omega, t) = (g^{1,j}, g^{2,j}, \dots)$'s are ℓ_2 -valued predictable processes such that each $g^k = (g^{k,1}, \dots, g^{k,m})$ is bounded. Then, under the assumption (3.2),

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^m} |g^k(s)|^2 |z|^2 N_k(s, dz) ds \right)^{p/2} \right] \\ & \leq c(p) \mathbb{E} \left[\left(\int_0^t \sum_{k=1}^{\infty} |g^k(s)|^2 ds \right)^{p/2} + \int_0^t \sum_{k=1}^{\infty} |g^k(s)|^p ds \right]. \end{aligned} \quad (3.6)$$

Proof. Due to monotone convergence theorem we may assume $g^{k,j} = 0$ for all $i > M$ and $1 \leq j \leq m$. By monotone convergence theorem,

$$\begin{aligned} A &:= \mathbb{E} \left[\left(\sum_{k=1}^M \int_0^t \int_{\mathbb{R}^m} |g^k(s)|^2 |z|^2 N_k(s, dz) ds \right)^{p/2} \right] \\ &= \lim_{N \rightarrow \infty} \mathbb{E} \left[\left(\sum_{k=1}^M \int_0^t \int_{|z| \leq N} |g^k(s)|^2 |z|^2 N_k(s, dz) ds \right)^{p/2} \right]. \end{aligned}$$

Since $(a + b)^{p/2} \leq c(p)(|a|^{p/2} + |b|^{p/2})$ and $\tilde{N}_k(s, dz) := N_k(s, dz) - s\nu_k(dz)$,

$$A \leq c(p) \lim_{N \rightarrow \infty} \mathbb{E} \left[(J_{2,t})^{p/2} \right] + c(p) \mathbb{E} \left[\left(\int_0^t \int_{\mathbb{R}^m} \sum_{k=1}^M |g^k(s)|^2 |z|^2 \nu_k(dz) ds \right)^{p/2} \right]$$

where $J_{n,t} := \sum_{k=1}^M \int_0^t \int_{|z| \leq N} |g^k(s)|^n |z|^n \tilde{N}_k(s, dz) ds$, which is a square integrable martingale because g^k are bounded predictable processes. By Burkholder-Davis-Gundy inequality (For example, see [22, Theorem 48].)

$$\begin{aligned} \mathbb{E} \left[(J_{2,t})^{p/2} \right] &\leq c(p) \mathbb{E} \left[[J_2]_t^{p/4} \right] = c(p) \mathbb{E} \left[\left(\sum_{k=1}^M \int_0^t \int_{|z| \leq N} |g^k(s)|^4 |z|^4 N_k(s, dz) ds \right)^{p/4} \right] \\ &\leq c(p) \mathbb{E} \left[\left(\sum_{k=1}^M \sum_{0 \leq s \leq t} |g^k(s)|^4 |\Delta Y_s^k|^4 \right)^{p/4} \right]. \end{aligned} \quad (3.7)$$

Recall that for any $q > 1$, $(\sum |a_n|^q)^{1/q} \leq \sum |a_n|$. Thus if $2 < p \leq 4$, then

$$\mathbb{E} \left[(J_{2,t})^{p/2} \right] \leq c(p) \mathbb{E} \left[\sum_{k=1}^M \sum_{0 \leq s \leq t} |g^k(s)|^p |\Delta Y_s^k|^p \right] \leq c(p, \hat{c}) \mathbb{E} \left[\int_0^t \sum_{k=1}^M |g^k(s)|^p ds \right].$$

If $4 < p \leq 8$ then, by the relation $\tilde{N}_k(s, dz) = N_k(s, dz) - s\nu_k(dz)$ and Burkholder-Davis-Gundy inequality,

$$\begin{aligned} \mathbb{E} \left[(J_{2,t})^{p/2} \right] &\leq c(p) \mathbb{E} \left[\left(\sum_{k=1}^M \int_0^t \int_{|z| \leq N} |g^k(s)|^4 |z|^4 N_k(s, dz) ds \right)^{p/4} \right] \\ &\leq c(p) \mathbb{E} \left[(J_{8,t})^{p/8} \right] + c(p) \mathbb{E} \left[\left(\sum_{k=1}^M \int_0^t \int_{|z| \leq N} |g(s)|^4 |z|^4 \nu_k(dz) ds \right)^{p/4} \right] \\ &\leq c(p, \hat{c}) \mathbb{E} \left[\left(\sum_{k=1}^M \int_0^t \int_{|z| \leq N} |g^k(s)|^8 |z|^8 N_k(s, dz) ds \right)^{p/8} + \left(\int_0^t \sum_{k=1}^M |g^k(s)|^4 ds \right)^{p/4} \right] \\ &\leq c(p, \hat{c}) \mathbb{E} \left[\left(\sum_{k=1}^M \int_0^t \int_{|z| \leq N} |g^k(s)|^p |z|^p N_k(s, dz) ds \right) + \left(\int_0^t \sum_{k=1}^M |g^k(s)|^4 ds \right)^{p/4} \right] \\ &\leq c(p, \hat{c}) \mathbb{E} \left[\int_0^t \sum_{k=1}^M |g^k(s)|^p ds + \left(\int_0^t \sum_{k=1}^M |g^k(s)|^4 ds \right)^{p/4} \right]. \end{aligned}$$

Similarly, in general, for $p \in (2^{n-1}, 2^n]$,

$$A \leq c(p, \hat{c}) \sum_{j=1}^n \mathbb{E} \left[\left(\int_0^t \sum_{k=1}^M |g^k(s)|^{2^j} ds \right)^{p^{2^{-j}}} \right] + c(p, \hat{c}) \mathbb{E} \left[\int_0^t \sum_{k=1}^M |g^k(s, x)|^p ds \right].$$

Also since for each $2 \leq q \leq p$,

$$\left(\int_0^t \sum_{k=1}^M |g^k(s)|^q ds \right)^{1/q} \leq \left(\left(\int_0^t \sum_{k=1}^M |g^k(s)|^2 ds \right)^{1/2} + \left(\int_0^t \sum_{k=1}^M |g^k(s)|^p ds \right)^{1/p} \right),$$

we get

$$A \leq c(p, \widehat{c}) \mathbb{E} \left[\left(\int_0^t \sum_{k=1}^\infty |g^k(s)|^2 ds \right)^{p/2} + \int_0^t \sum_{k=1}^\infty |g^k(s)|^p ds \right]. \quad (3.8)$$

Thus the lemma is proved. \square

Theorem 3.4 Suppose that (3.2) holds. For any $p \in [2, \infty)$ and $\gamma \in \mathbb{R}$, $\mathcal{H}_p^{\gamma+\alpha}(T)$ is a Banach space with norm $\|\cdot\|_{\mathcal{H}_p^{\gamma+\alpha}(T)}$. Moreover, there is a constant $c = c(d, p, T) > 0$ such that for every $u \in \mathcal{H}_p^{\gamma+2}(T)$ and $0 < t \leq T$,

$$\mathbb{E} \left[\sup_{s \leq t} \|u(s, \cdot)\|_{H_p^\gamma}^p \right] \leq c(p, d, T) \left(\|\mathbb{D}u\|_{\mathbb{H}_p^\gamma(t)}^p + \|\mathbb{S}_c u\|_{\mathbb{H}_p^\gamma(t, \ell_2)}^p + \sum_{j=1}^m \|\mathbb{S}_d^{\cdot j} u\|_{\mathbb{H}_p^\gamma(t, \ell_2)}^p + \|u_0\|_{U_p^\gamma}^p \right). \quad (3.9)$$

Proof. By Theorem 2.6 and the reasons explained just before Remark 3.1, without loss of generality we assume that $Y_t^k = \int_{\mathbb{R}^m} z \tilde{N}_k(t, dz)$. Moreover, due to Remark 2.8 it suffices to prove the theorem only for $\gamma = 0$. First we prove (3.9). Let $du = f dt + \sum_{k=1}^\infty g^k \cdot dY_t^k$ with $u(0) = u_0$.

For a moment, we assume that $g^{k,j} = 0$ for all $k \geq N_0, 1 \leq j \leq m$ and $g^{k,j}$ is of the type

$$g^{k,j}(t, x) = \sum_{i=0}^{m_k} I_{(\tau_i^{k,j}, \tau_{i+1}^{k,j}]}(t) g^{k_i,j}(x), \quad (3.10)$$

where $\tau_i^{k,j}$ are bounded stopping times and $g^{k_i,j} \in C_0^\infty(\mathbb{R}^d)$. Define

$$v(t, x) = \sum_{k=1}^{N_0} \int_0^t g^k(s, x) \cdot dY_s^k.$$

Then by Burkholder-Davis-Gundy inequality and Lemma 3.3,

$$\begin{aligned} \mathbb{E} \left[\sup_{s \leq t} |v(s, x)|^p \right] &= \mathbb{E} \left[\sup_{s \leq t} \left| \sum_{k=1}^{N_0} \sum_{j=1}^m \int_0^s \int_{\mathbb{R}^m} g^{k,j}(s, x) z^j \tilde{N}_k(s, dz) ds \right|^p \right] \\ &\leq c(p) \mathbb{E} \left[\left(\sum_{k=1}^\infty \sum_{i,j=1}^m \int_0^t \int_{\mathbb{R}^m} g^{k,i}(s, x) g^{k,j}(s, x) z^i z^j N_k(s, dz) ds \right)^{p/2} \right] \\ &\leq c(p, \widehat{c}) \mathbb{E} \left[\left(\sum_{k=1}^\infty \int_0^t \int_{\mathbb{R}^m} |g^k(s, x)|^2 |z|^2 N_k(s, dz) ds \right)^{p/2} \right] \\ &\leq c(p, \widehat{c}) \mathbb{E} \left[\left(\int_0^t \sum_{k=1}^\infty |g^k(s, x)|^2 ds \right)^{p/2} + \int_0^t \sum_{k=1}^\infty |g^k(s, x)|^p ds \right]. \end{aligned}$$

Since $\sum_n |a_n|^p \leq (\sum_n |a_n|^2)^{p/2}$ and $(\int_0^t |f| ds)^p \leq t^{p-1} \int_0^t |f|^p ds$, by integrating over \mathbb{R}^d we get that for every $t \leq T$

$$\mathbb{E} \left[\sup_{s \leq t} \|v\|_p^p \right] \leq c(T, p) \sum_{j=1}^m \|g^{\cdot, j}\|_{\mathbb{L}_p(t, \ell_2)}^p := c(T, p) \sum_{j=1}^m \mathbb{E} \int_0^t \int_{\mathbb{R}^d} |g^{\cdot, j}|_{\ell_2}^p dx ds. \quad (3.11)$$

Next we prove (3.11) for general $g^{\cdot, j} \in \mathbb{L}_p(T, \ell_2)$. By Theorem 3.10 in [14], we can take a sequence $g_n^{\cdot, j} \in \mathbb{L}_p(T, \ell_2)$ so that for each fixed n , $g_n^{k, j} = 0$ for all large k and each $g_n^{k, j}$ is of the type (3.10), and $g_n^{\cdot, j} \rightarrow g^{\cdot, j}$ in $\mathbb{L}_p(T, \ell_2)$ as $n \rightarrow \infty$. Define $v_n(t, x) = \sum_{k=1}^\infty \sum_{j=1}^m \int_0^t g_n^{k, j} dY_t^{k, j}$, then for every $t \leq T$

$$\mathbb{E} \left[\sup_{s \leq t} \|v_n\|_p^p \right] \leq c(T, p) \sum_{j=1}^m \|g_n^{\cdot, j}\|_{\mathbb{L}_p(t, \ell_2)}^p, \quad \mathbb{E} \left[\sup_{s \leq t} \|v_{n_1} - v_{n_2}\|_p^p \right] \leq c(T, p) \sum_{j=1}^m \|g_{n_1}^{\cdot, j} - g_{n_2}^{\cdot, j}\|_{\mathbb{L}_p(t, \ell_2)}^p.$$

Thus (3.11) follows by taking $n \rightarrow \infty$. Now note that

$$d(u - v) = f dt \quad \text{with} \quad (u - v)(0) = u_0.$$

Thus it is easy to check that

$$\mathbb{E} \left[\sup_{s \leq t} \|u - v\|_p^p \right] \leq N \mathbb{E} [\|u_0\|_p^p] + N \mathbb{E} \left[\int_0^t \|f(s, \cdot)\|_p^p ds \right].$$

Consequently,

$$\mathbb{E} \left[\sup_{s \leq t} \|u\|_p^p \right] \leq N \|f\|_{\mathbb{H}_p^\alpha(t)}^p + c \sum_{j=1}^m \|g^{\cdot, j}\|_{\mathbb{L}_p(t, \ell_2)}^p + N \mathbb{E} \|u_0\|_p^p.$$

The completeness of the space $\mathcal{H}_p^\alpha(T)$ easily follows from (3.9). Indeed, let $\{u_n : n = 1, 2, \dots\}$ be a Cauchy sequence in $\mathcal{H}_p^\alpha(T)$. Then $\{u_n\}$, $\{\mathbb{D}u_n\}$, $\{\mathbb{S}_d^{\cdot, j} u_n\}$ and $\{u_n(0)\}$ are Cauchy sequences in $\mathbb{H}_p^\alpha(T)$, $\mathbb{H}_p^{\alpha/2}(T, \ell_2)$ and $U_p^{\alpha/2-\alpha/p}$ respectively. Thus there exist $u \in \mathbb{H}_p^\alpha(T)$, $f \in \mathbb{L}_p(T)$, $g^{\cdot, j} \in \mathbb{H}_p^{\alpha/2}(T, \ell_2)$ and $u_0 \in U_p^{\alpha/2-\alpha/p}$ so that $u_n, \mathbb{D}u_n, \mathbb{S}_d^{\cdot, j} u_n, u_n(0)$ converge to $u, f, g^{\cdot, j}, u_0$ respectively, that is,

$$\|u_n - u\|_{\mathbb{H}_p^\alpha(T)} + \|\mathbb{D}u_n - f\|_{\mathbb{L}_p(T)} + \|\mathbb{S}_d^{\cdot, j} u_n - g^{\cdot, j}\|_{\mathbb{H}_p^{\alpha/2}(T, \ell_2)} + \|u_n(0) - u_0\|_{U_p^{\alpha/2-\alpha/p}} \rightarrow 0$$

as $n \rightarrow \infty$. Thus to prove $u \in \mathcal{H}_p^\alpha(T)$ and $u_n \rightarrow u$ in $\mathcal{H}_p^\alpha(T)$, we only need to show that for any $\phi \in C_0^\infty(\mathbb{R}^d)$, the equality

$$(u(t, \cdot), \phi) = (u_0, \phi) + \int_0^t (f(s, \cdot), \phi) ds + \sum_{k=1}^\infty \sum_{j=1}^m \int_0^t (g^{k, j}(s, \cdot), \phi) dY_s^{k, j} \quad (3.12)$$

holds for all $t \leq T$ (a.s.). Taking the limit from

$$(u_n(t, \cdot), \phi) = (u_n(0), \phi) + \int_0^t (\mathbb{D}u_n(s, \cdot), \phi) ds + \sum_{k=1}^\infty \sum_{j=1}^m \int_0^t (\mathbb{S}_d^{k, j} u_n(s, \cdot), \phi) dY_s^{k, j}$$

and using the argument used in Remark 3.1(ii) one can show that (3.12) holds in $\Omega \times [0, T]$ (a.e.). Also using the inequality (see (3.9))

$$\mathbb{E} \left[\sup_{t \leq T} \|u_n(\cdot, t) - u_m(\cdot, t)\|_{L_p}^p \right] \leq N \|u_n - u_m\|_{\mathcal{H}_p^\alpha(T)}$$

and taking $m \rightarrow \infty$, one finds that $(u(t, \cdot), \phi)$ is right continuous with left limits, and consequently (3.12) holds for all $t \leq T$ (a.s.). The theorem is proved. \square

Lemma 3.5 *Let $p \in (2, \infty)$, $t > 0$ and $f \in L_p([0, t] \times \mathbb{R}^d)$. Then for any $\varepsilon > \alpha(1/2 - 1/p)$,*

$$\int_{\mathbb{R}^d} \int_0^t \int_0^s |\partial_x^{\alpha/2} T_{s-r} f(r, x)|^p dr ds dx \leq c \int_0^t \|f(s, \cdot)\|_{H_p^\varepsilon}^p ds, \quad (3.13)$$

where $c = c(d, p, \alpha, \varepsilon)$ is independent of t .

Proof. Note that we may assume $\alpha(1/2 - 1/p) < \varepsilon < \alpha/2$. Let $q > p$ be chosen so that

$$\frac{1}{p} = \left(1 - \frac{2\varepsilon}{\alpha}\right) \times \frac{1}{2} + \frac{2\varepsilon}{\alpha} \times \frac{1}{q}.$$

Such choice of q is possible since $1/p > (1 - \frac{2\varepsilon}{\alpha}) \times \frac{1}{2}$. We will use an interpolation theorem. First, note that

$$\varepsilon = \left(1 - \frac{2\varepsilon}{\alpha}\right) \times 0 + \frac{2\varepsilon}{\alpha} \times \frac{\alpha}{2}.$$

Define an operator \mathcal{A} by

$$\mathcal{A}f(s, r, x) = \begin{cases} \partial^{\alpha/2} T_{s-r} f & \text{if } r < s, \\ 0 & \text{otherwise.} \end{cases}$$

Then, due to (2.19) and the inequality $\|T_{s-r} \partial^{\alpha/2} f\|_q \leq \|\partial^{\alpha/2} f\|_q \leq \|f\|_{H_q^{\alpha/2}}$, the linear mappings

$$\mathcal{A} : L_2([0, t], L_2(\mathbb{R}^d)) \rightarrow L_2([0, t] \times [0, t] \times \mathbb{R}^d)$$

and

$$\mathcal{A} : L_q([0, t], H_q^{\alpha/2}) \rightarrow L_q([0, t] \times [0, t] \times \mathbb{R}^d)$$

are bounded and their norms are independent of t . It follows from the interpolation theory (see, for instance, [3, Theorem 5.1.2]) that the operator

$$\mathcal{A} : L_p([0, t], H_p^\varepsilon(\mathbb{R}^d)) \rightarrow L_p([0, t] \times [0, t] \times \mathbb{R}^d)$$

is bounded and its norm is independent of t . The lemma is proved. \square

Theorem 3.6 Fix a constant ε_1 so that $\varepsilon_1 = 0$ if $p = 2$, and $\varepsilon_1 > \alpha(1/2 - 1/p)$ if $p > 2$. suppose (3.2) holds. Then for any $f \in \mathcal{H}_p^\gamma(T)$, $h \in \mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)$, $g^{:,j} \in \mathcal{H}_p^{\gamma+\alpha/2+\varepsilon_1}(T, \ell_2)$, $1 \leq j \leq m$ and $u_0 \in U_p^{\gamma+\alpha/2-\alpha/p}$, equation (3.4) has a unique solution u in $\mathcal{H}_p^{\gamma+\alpha}(T)$, and for this solution

$$\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(t)} \leq c(p, T, \delta) \left(\|f\|_{\mathbb{H}_p^\gamma(t)} + \|h\|_{\mathbb{H}_p^{\gamma+\alpha/2}(t, \ell_2)} + \sum_{j=1}^m \|g^{:,j}\|_{\mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(t, \ell_2)} + \|u_0\|_{U_p^{\gamma+\alpha-\alpha/p}} \right) \quad (3.14)$$

for every $t \leq T$.

Proof. As explained before, without loss of generality we assume that h^i 's are all zeros.

Step 1. As in the proof of Theorem 2.11, we only need to prove the theorem for a particular $\gamma = \gamma_0$.

Step 2. We assume $a(\omega) = 1$ and prove the theorem for the equation:

$$du = \Delta^{\alpha/2} u dt + \sum_{k=1}^{\infty} g^k dY_t^k, \quad u(0) = 0. \quad (3.15)$$

By the result of Step 1, we may assume that $\gamma = -\alpha/2$. The uniqueness is obvious and we only prove the existence and the estimate (3.14). Considering approximation arguments, for a moment, we assume that $g^{k,j} = 0$ for all $k > N_0$ and $1 \leq j \leq m$ and that

$$g^{k,j}(t, x) = \sum_{i=0}^{m_k} I_{(\tau_i^{k,j}, \tau_{i+1}^{k,j}]}(t) g^{k_i,j}(x),$$

where $\tau_i^{k,j}$ are bounded stopping times and $g^{k_i,j}(x) \in C_0^\infty(\mathbb{R}^d)$. Define

$$v(t, x) := \sum_{k=1}^{N_0} \int_0^t g^k(s, x) \cdot dY_s^k = \sum_{k=1}^{N_0} \sum_{i=1}^{m_k} \sum_{j=1}^m g^{k_i,j}(x) (Y_{t \wedge \tau_{i+1}^k}^{k,j} - Y_{t \wedge \tau_i^k}^{k,j})$$

and

$$u(t, x) := v(t, x) + \int_0^t \Delta^{\alpha/2} T_{t-s} v ds = v(t, x) + \int_0^t T_{t-s} \Delta^{\alpha/2} v ds. \quad (3.16)$$

Now we remember from the proof of Theorem 2.11 that if functions $h_1 = h_1(t, x)$ and $h_2 = h_2(x)$ are sufficiently smooth, then

$$w_1(t, x) := \int_0^t T_{t-s} h_1(s) ds, \quad w_2(t, x) = T_t h_2$$

solve

$$\begin{aligned} dw_1 &= (\Delta^{\alpha/2} w + h_1) dt, \quad w_1(0) = 0, \\ dw_2 &= \Delta^{\alpha/2} w_2 dt, \quad w_2(0) = h_2. \end{aligned}$$

Therefore we have $d(u - v) = (\Delta^{\alpha/2}(u - v) + \Delta^{\alpha/2}v)dt = \Delta^{\alpha/2}u dt$, and

$$du = \Delta^{\alpha/2}u dt + dv = \Delta^{\alpha/2}u dt + \sum_{k=1}^{N_0} g^k \cdot dY_t^k.$$

Let $T_{t-s}g^k(r, x) = (T_{t-s}g^{k,1}(r, x), \dots, T_{t-s}g^{k,m}(r, x))$. By (3.16) and stochastic Fubini theorem ([22, Theorem 64]), almost surely,

$$\begin{aligned} u(t, x) &= v(t, x) + \sum_{k=1}^{N_0} \int_0^t \int_0^s \Delta^{\alpha/2} T_{t-s} g^k(r, x) \cdot dY_r^k ds \\ &= v(t, x) - \sum_{k=1}^{N_0} \sum_{j=1}^m \int_0^t \int_r^t \frac{\partial}{\partial s} T_{t-s} g^{k,j}(r, x) ds dY_r^{k,j} \\ &= \sum_{k=1}^{N_0} \int_0^t T_{t-s} g^k(s, x) \cdot dY_s^k. \end{aligned} \tag{3.17}$$

Hence,

$$\partial_x^{\alpha/2} u(t, x) = \sum_{k=1}^{N_0} \int_0^t \partial_x^{\alpha/2} T_{t-s} g^k(s, x) \cdot dY_s^k = \sum_{k=1}^{N_0} \sum_{j=1}^m \int_0^t \partial_x^{\alpha/2} T_{t-s} g^{k,j}(s, x) dY_s^{k,j}.$$

By Burkholder-Davis-Gundy's inequality and Lemma 3.3, we have for every $0 < t \leq T$

$$\begin{aligned} \mathbb{E} \left[|\partial_x^{\alpha/2} u(t, x)|^p \right] &\leq c(p) \mathbb{E} \left[\left(\sum_{k=1}^{N_0} \sum_{i,j=1}^m \int_0^t \int_{\mathbb{R}^m} \partial_x^{\alpha/2} T_{t-s} g^{k,i}(s, x) \partial_x^{\alpha/2} T_{t-s} g^{k,j}(s, x) z^i z^j N^k(dz, ds) \right)^{p/2} \right] \\ &\leq c(p) \mathbb{E} \left[\left(\sum_{k=1}^{N_0} \int_0^t \int_{\mathbb{R}^m} |\partial_x^{\alpha/2} T_{t-s} g^k(s, x)|^2 |z|^2 N^k(dz, ds) \right)^{p/2} \right] \\ &\leq c(p) \mathbb{E} \left[\left(\int_0^t \sum_{k=1}^{\infty} |\partial_x^{\alpha/2} T_{t-s} g^k(s, x)|^2 ds \right)^{p/2} + \int_0^t \sum_{k=1}^{\infty} |\partial_x^{\alpha/2} T_{t-s} g^k(s, x)|^p ds \right]. \end{aligned}$$

By (2.19), Lemma 3.5 and the inequality $\sum_{k=1}^{\infty} |a_k|^p \leq (\sum_{k=1}^{\infty} |a_n|^2)^{p/2}$,

$$\mathbb{E} \left[\int_0^t \|\partial_x^{\alpha/2} u(s, \cdot)\|_p^p ds \right] \leq c(p, \alpha) \sum_{j=1}^m \mathbb{E} \left[\int_0^t \|g^{\cdot,j}(s, \cdot)\|_{H_p^{\varepsilon_1}(\ell_2)}^p dt \right]. \tag{3.18}$$

Similarly from (3.17) we also get, for every $0 < t \leq T$,

$$\mathbb{E} [|u(t, x)|^p] \leq c(p) \mathbb{E} \left[\left(\int_0^t \sum_{k=1}^{N_0} |T_{t-s} g^k(s, x)|^2 ds \right)^{p/2} + \int_0^t \sum_{k=1}^{N_0} |T_{t-s} g^k(s, x)|^p ds \right]. \tag{3.19}$$

By the same argument which leads to (2.22), we see that the right side of (3.19) is finite. Thus we proved $\partial_x^{\alpha/2} u, u \in \mathbb{L}_p(T)$, and hence $u \in \mathcal{H}_p^{\alpha/2}(T)$. As in (2.23) and (2.24),

$$\begin{aligned}
\|u\|_{\mathcal{H}_p^{\alpha/2}(t)}^p &\leq c(p) \left(\|u\|_{\mathbb{H}_p^{\alpha/2}(t)}^p + \|\Delta^{\alpha/2} u\|_{\mathbb{H}_p^{-\alpha/2}(t)}^p + \|g\|_{\mathbb{L}_p(t, \ell_2)}^p \right) \\
&\leq c \left(\|u\|_{\mathbb{H}_p^{-\alpha/2}(t)}^p + \|\partial_x^{\alpha/2} u\|_{\mathbb{L}_p(t)}^p + \|g\|_{\mathbb{L}_p(t, \ell_2)}^p \right) \\
&\leq c(p, T, \alpha) \left(\|u\|_{\mathbb{H}_p^{-\alpha/2}(t)}^p + \|g\|_{\mathbb{H}_p^{\varepsilon_1}(t, \ell_2)}^p \right) \\
&\leq c(p, T, \alpha) \int_0^t \|u\|_{\mathcal{H}_p^{\alpha/2}(s)}^p ds + c(p, T, \alpha) \|g\|_{\mathbb{H}_p^{\varepsilon_1}(T, \ell_2)}^p.
\end{aligned}$$

Finally, Gronwall leads to (3.14). Once one has a unique solvability of equation (3.15) and estimate (3.14) for sufficiently smooth g , we repeat the same approximation argument used in the Step 2 of the proof of Theorem 2.11.

Step 3. Now, we follow Step 3–Step 5 of the proof of Theorem 2.11 word for word except obvious changes from W_t^k to Y_t^k . The theorem is proved. \square

Finally we consider the nonlinear equation

$$du = \left(a(\omega, t) \Delta^{\alpha/2} u + f(u) \right) dt + \sum_{k=1}^{\infty} h^k(u) dW_t^i + \sum_{k=1}^{\infty} \sum_{j=1}^m g^{k,j}(u) dY_t^{k,j}, \quad u(0) = u_0, \quad (3.20)$$

where $f(u) = f(\omega, t, x, u)$, $h^k(u) = h^k(\omega, t, x, u)$, $g^k(u) = (g^{k,1}(\omega, t, x, u), \dots, g^{k,m}(\omega, t, x, u))$, W_t are independent 1-dimensional Wiener processes and $Y_t^k := \int_{\mathbb{R}^m} z \tilde{N}_k(t, dz)$ are independent m -dimensional pure jump Lévy processes with Lévy measure ν_k .

Assumption 3.7 Fix a constant ε_1 so that $\varepsilon_1 = 0$ if $p = 2$, and $\varepsilon_1 > \alpha(1/2 - 1/p)$ if $p > 2$. Assume that $f(0) \in \mathbb{H}_p^{\gamma}(T)$, $g^{j,j}(0) \in \mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(T, \ell_2)$ and $h(0) \in \mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)$. Moreover, for any $\varepsilon > 0$, there exists a constant K_{ε} so that for any $u = u(x), v = v(x) \in H_p^{\gamma+\alpha}$ and t, ω we have

$$\begin{aligned}
&\|f(t, \cdot, u(\cdot)) - f(t, \cdot, v(\cdot))\|_{H_p^{\gamma}} + \sum_{j=1}^m \|g^{j,j}(t, \cdot, u(\cdot)) - g^{j,j}(t, \cdot, v(\cdot))\|_{H_p^{\gamma+\alpha/2+\varepsilon_1}(\ell_2)} \\
&+ \|h(t, \cdot, u(\cdot)) - h(t, \cdot, v(\cdot))\|_{H_p^{\gamma+\alpha/2}(\ell_2)} \leq \varepsilon \|u - v\|_{H_p^{\gamma+\alpha}} + K(\varepsilon) \|u - v\|_{H_p^{\gamma}}.
\end{aligned} \quad (3.21)$$

Example 3.8 Recall that the space B^r is defined in (2.35). Fix $\kappa_0 = \kappa_0(\gamma) \geq 0$ so that $\kappa_0 > 0$ if γ is not integer. Consider

$$\begin{aligned}
f(u) &= b(\omega, t, x) \Delta^{\beta_1/2} u + \sum_{i=1}^d c^i(\omega, t, x) u_{x^i} I_{\alpha>1} + d(\omega, t, x) u + f_0, \\
h^k(u) &= \eta^k(\omega, t, x) \Delta^{\beta_2/2} u + l^k(\omega, t, x) u + h_0^k, \\
g^{k,j}(u) &= \sigma^{k,j}(\omega, t, x) \Delta^{\beta_3/2} u + v^{k,j}(\omega, t, x) u + g_0^{k,j}, \quad j = 1, \dots, m.
\end{aligned}$$

Here $\beta_1 < \alpha$, $\beta_2 < \alpha/2$, $\beta_3^j < \alpha/2 - \varepsilon_1$ and $f_0 \in \mathbb{H}_p^\gamma(T)$, $h_0 \in \mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)$, $g_0^{\cdot,j} \in \mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(T, \ell_2)$. Assume for each ω, t, i, j ,

$$\begin{aligned} & |b|_{B^{|\gamma|+\kappa_0}} + |c^i|_{B^{|\gamma|+\kappa_0}} + |d|_{B^{|\gamma|+\kappa_0}} + |\eta|_{B^{|\gamma|+\alpha/2+\kappa_0}} + |l|_{B^{|\gamma|+\alpha/2+\kappa_0}} \\ & + |\sigma^{\cdot,j}|_{B^{|\gamma|+\alpha/2+\varepsilon_1+\kappa_0}} + |v^{\cdot,j}|_{B^{|\gamma|+\alpha/2+\varepsilon_1+\kappa_0}} \leq K < \infty. \end{aligned}$$

Then the calculus in Example 2.14 shows that (3.21) holds.

Here is the main result of this section.

Theorem 3.9 *Suppose (3.2) and Assumptions 2.10 and 3.7 hold. Then the equation (3.20) has a unique solution $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$, and for this solution we have*

$$\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(t)} \leq c \left(\|f(0)\|_{\mathbb{H}_p^\gamma(t)} + \|h(0)\|_{\mathbb{H}_p^{\gamma+\alpha/2}(t, \ell_2)} + \sum_{j=1}^m \|g^{\cdot,j}(0)\|_{\mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(t, \ell_2)} + \|u_0\|_{U_p^{\gamma+\alpha/2-\alpha/p}} \right), \quad (3.22)$$

for every $t \leq T$, where $c = c(p, T, \delta)$.

Proof. As we mentioned in the previous section, our proof is a repetition of that of Theorem 6.4 in [14]. By Theorem 3.6, for any $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$ with initial data u_0 we can define $v = \mathcal{R}u$ as the solution of

$$dv = \left(a(\omega, t) \Delta^{\alpha/2} v(t, x) + f(t, x, u) \right) dt + \sum_{i=1}^{\infty} h^i(t, x, u) dW_t^i + \sum_{k=1}^{\infty} \sum_{j=1}^m g^{k,j}(t, x, u) dY_t^{k,j}, \quad v(0) = u_0.$$

Then for any u, v initial data u_0 , we have $(\mathcal{R}u - \mathcal{R}v)(0, x) = 0$ and

$$\begin{aligned} d(\mathcal{R}u - \mathcal{R}v) &= \left(a(\omega, t) \Delta^{\alpha/2} (\mathcal{R}u - \mathcal{R}v) + (f(t, x, u) - f(t, x, v)) \right) dt \\ &\quad + \sum_{i=1}^{\infty} \int_0^t (h^i(t, x, u) - h^i(t, x, v)) dW_t^i + \sum_{k=1}^{\infty} \sum_{j=1}^m \int_0^t (g^{k,j}(t, x, u) - g^{k,j}(t, x, v)) dY_t^{k,j}. \end{aligned}$$

By Theorems 2.11 and Assumption 3.7, for every $t \in (0, T]$,

$$\begin{aligned} & \|\mathcal{R}u - \mathcal{R}v\|_{\mathcal{H}_p^{\gamma+\alpha}(t)}^p \\ & \leq c(p, T, \delta) \left(\|f(u) - f(v)\|_{\mathbb{H}_p^\gamma(t)}^p + \|h(u) - h(v)\|_{\mathbb{H}_p^{\gamma+\alpha/2}(t, \ell_2)}^p + \sum_{j=1}^m \|g^{\cdot,j}(u) - g^{\cdot,j}(v)\|_{\mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(t, \ell_2)}^p \right) \\ & \leq \varepsilon^p c(p, T, \delta) \|u - v\|_{\mathcal{H}_p^{\gamma+\alpha}(t)}^p + K(\varepsilon) c(p, T, \delta) \int_0^t \mathbb{E} \|u(s, \cdot) - v(s, \cdot)\|_{H_p^\gamma}^p ds \\ & \leq \theta \|u - v\|_{\mathcal{H}_p^{\gamma+\alpha}(t)}^p + N \int_0^t \|u - v\|_{\mathcal{H}_p^{\gamma+\alpha}(s)}^p ds \end{aligned}$$

where $\theta := \varepsilon^p c(p, T, \delta)$ and $N = c(p, T, \delta, \varepsilon)$. Denote $\mathcal{R}^{n+1}u := \mathcal{R}(\mathcal{R}^n u)$. Then by induction, for every $t \in (0, T]$

$$\begin{aligned} & \|\mathcal{R}^n u - \mathcal{R}^n v\|_{\mathcal{H}_p^{\gamma+\alpha}(t)}^p \\ & \leq \theta^n \|u - v\|_{\mathcal{H}_p^{\gamma+\alpha}(t)}^p + \sum_{k=1}^n \binom{n}{k} \theta^{n-k} N^k \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} \|u - v\|_{\mathcal{H}_p^{\gamma+\alpha}(s)}^p ds \end{aligned}$$

Therefore,

$$\begin{aligned} \|\mathcal{R}^n u - \mathcal{R}^n v\|_{\mathcal{H}_p^{\gamma+\alpha}(T)}^p & \leq \theta^n \sum_{k=0}^n \binom{n}{k} \frac{(NT/\theta)^k}{k!} \|u - v\|_{\mathcal{H}_p^{\gamma+\alpha}(T)}^p \\ & \leq (2\theta)^n \left(\sup_{k \geq 0} \frac{(NT/\theta)^k}{k!} \right) \|u - v\|_{\mathcal{H}_p^{\gamma+\alpha}(T)}^p. \end{aligned}$$

Choose $\varepsilon > 0$ so that $2\theta < 1/2$, and then fix n large enough so that $(2\theta)^n \left(\sup_{k \geq 0} \frac{(NT/\theta)^k}{k!} \right) < 1/2$. Then $\bar{\mathcal{R}} := \mathcal{R}^n$ is a contraction in $\mathcal{H}_p^{\gamma+\alpha}(T)$ and obviously the unique fixed point u under this map becomes the unique solution of (3.20). Moreover, the estimate (3.22) also easily from Assumption 3.7, Theorems 3.6 and 3.4. We leave the details to the readers as an exercise. \square

4 Application and Extension

First, we consider equations with the random fractional Laplacian driven by (Lévy) space-time white noise; Let $d = 1$ and consider the equation

$$du = (a(\omega, t) \Delta^{\alpha/2} u(t, x) + f(\omega, t, x, u(t, x))) dt + \xi(\omega, t, x) h(\omega, t, x, u(t, x)) d\mathcal{Z}_t \quad (4.23)$$

where \mathcal{Z}_t is a cylindrical Lévy process on $L_2(\mathbb{R})$, that is \mathcal{Z}_t has an expansion of the form

$$\mathcal{Z}_t = \sum_{k=1}^{\infty} \eta^k(x) Z_t^k$$

where $\{\eta^k : k = 1, 2, \dots\}$ is an orthonormal basis in L_2 and Z_t^k are i.i.d. one-dimensional \mathcal{F}_t -adapted Lévy processes (see [10] for the details). Using this expansion we can rewrite (4.23) as follows :

$$du = (a(\omega, t) \Delta^{\alpha/2} u + f(u)) dt + \sum_{k=1}^{\infty} g^k(u) dZ_t^k, \quad (4.24)$$

where $g^k(u) := \xi(\omega, t, x) h(\omega, t, x, u(t, x)) \eta^k(x)$.

Let γ, p, s, r be constants satisfying

$$0 > \gamma + \alpha/2 > -1, \quad p \geq 2r \geq 2, \quad 1 \leq r < (2\gamma + \alpha + 2)^{-1}, \quad s^{-1} + r^{-1} = 1 \quad (1 \leq s \leq \infty). \quad (4.25)$$

Define

$$R_\gamma(x) := |x|^{-(\gamma+\alpha/2+1)} \int_0^\infty t^{-(\gamma+\alpha/2+3)/2} e^{-tx^2-1/(4t)} dt.$$

It is known that there exists a constant $c > 0$ so that $cR_\gamma(x)$ is the kernel of the operator $(1 - \Delta)^{(\gamma+\alpha/2)/2}$, that is $(1 - \Delta)^{(\gamma+\alpha/2)/2} f = (cR_\gamma * f)(x)$.

Assumption 4.1 (i) For each x , $\xi = \xi(\omega, t, x)$ is predictable, and $\|\xi(\omega, t, \cdot)\|_{L_{2s}} \leq K$ for each ω, t .
(ii) For each x, u , the processes $f(\omega, t, x, u), h(\omega, t, x, u)$ are predictable, and

$$|f(\omega, t, x, u) - f(\omega, t, x, v)| \leq K|u - v|, \quad |h(\omega, t, x, u) - h(\omega, t, x, v)| \leq K|u - v|.$$

By following the arguments in the proof of [14, Lemma 8.4], we get the following

Lemma 4.2 Let (4.25) hold. Take some functions $h_0 = h_0(x) \in L_p(\mathbb{R})$, $\xi_0 = \xi_0(x) \in L_{2s}(\mathbb{R})$, and set $g_0^k = \xi_0 h_0 \eta^k$. Then $g_0 = \{g_0^k\} \in H_p^{\gamma+\alpha/2}(\ell_2)$ and

$$\|g_0\|_{H_p^{\gamma+\alpha/2}(\ell_2)} = \|\bar{h}_{0,\gamma}\|_p \leq N \|\xi_0\|_{2s} \|h_0\|_p,$$

where $N = \|R_\gamma\|_{2r} < \infty$ and

$$\bar{h}_{0,\gamma}(x) := \left(\int_{\mathbb{R}} R_\gamma^2(x-y) \xi_0^2(y) h_0^2(y) dy \right)^{1/2}.$$

We first discuss the case when Z_t^k are independent one-dimensional Wiener processes.

Theorem 4.3 Let Z_t^k be independent one-dimensional Wiener processes. Suppose (4.25) and Assumption 4.1 hold. Also assume $\gamma \in (-\alpha, \frac{-1-\alpha}{2})$, $u_0 \in U_p^{\gamma+\alpha-\alpha/p}$ and

$$I(p, T) := \left(\mathbb{E} \int_0^T \left(\|f(t, \cdot, 0)\|_{H_p^\gamma}^p + \|\bar{h}(t, \cdot, 0)\|_p^p \right) ds \right)^{1/p} < \infty, \quad (4.26)$$

where

$$\bar{h}(t, x, 0) := \left(\int_{\mathbb{R}} R_\gamma^2(x-y) \xi^2(y) h^2(t, y, 0) dy \right)^{1/2}.$$

Then equation (4.23) with initial data u_0 has a unique solution $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$ and for this solution,

$$\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(T)} \leq c \left(I(p, T) + \|u_0\|_{U_p^{\gamma+\alpha-\alpha/p}} \right).$$

Proof. We check whether $f(u)$ and $g(u)$ satisfy condition (2.34). Since $\gamma < 0$ and $\gamma + \alpha > 0$,

$$\|f(u) - f(v)\|_{H_p^\gamma} \leq \|f(u) - f(v)\|_{L_p} \leq K\|u - v\|_{L_p} \leq \varepsilon\|u - v\|_{H_p^{\gamma+\alpha}} + K(\varepsilon)\|u - v\|_{H_p^\gamma}.$$

Also for $g(u) = \{g^k(u)\}$, by Lemma 4.2,

$$\|g(0)\|_{H_p^{\gamma+\alpha/2}(\ell_2)} \leq \|R_\gamma\|_{2r} \|\xi\|_{2s} \|h(0)\|_p \leq c \|h(0)\|_p,$$

$$\|g(u) - g(v)\|_{H_p^{\gamma+\alpha/2}(\ell_2)} \leq \|R_\gamma\|_{2r\xi}\|_{2s}\|h(u) - h(v)\|_p \leq c\|u - v\|_{L_p} \leq \varepsilon\|u - v\|_{H_p^{\gamma+\alpha}} + K(\varepsilon)\|u - v\|_{H_p^\gamma}.$$

Therefore condition (2.34) is satisfied and the theorem is proved. \square

Now we consider space-time white noise with jump Lévy processes. Unlike Theorem 4.3, in the case space-time white noise with jump Lévy processes, L_p -theory is not satisfactory due to the condition $\varepsilon_1 > \alpha(1/2 - 1/p)$ if $p > 2$. Thus we only give an L_2 -theory.

Theorem 4.4 *Suppose Z_t^k are independent one-dimensional jump Lévy processes with Lévy measure ν . Suppose (3.2), (4.25) and Assumption 4.1 hold with $p = 2$. Also assume $\gamma \in (-\alpha, \frac{-1-\alpha}{2})$, $u_0 \in U_2^{\gamma+\alpha-\alpha/2}$ and $I(2, T) < \infty$, where $I(2, T)$ is taken from (4.26). Then equation (4.23) with initial data u_0 has a unique solution $u \in \mathcal{H}_2^{\gamma+\alpha}(T)$ and for this solution,*

$$\|u\|_{\mathcal{H}_2^{\gamma+\alpha}(T)} \leq c \left(I(2, T) + \|u_0\|_{U_2^{\gamma+\alpha-\alpha/2}} \right).$$

Proof. There is nothing to prove since conditions on f and g were already checked in the proof of Theorem 4.3. \square

For a stopping time τ relative to $\{\mathcal{F}_t\}$, denote

$$[0, \tau] := \{(\omega, t) : 0 < t \leq \tau(\omega)\}.$$

Then obviously the process $\mathbf{1}_{[0, \tau]}(\omega, t)$ is left-continuous and predictable. For an H_p^γ -valued $\mathcal{P}^{dP \times dt}$ -measurable process u , write $u \in \mathbb{H}_p^\gamma(\tau)$ if

$$\|u\|_{\mathbb{H}_p^\gamma(\tau)}^2 := \mathbb{E} \left[\int_0^\tau \|u\|_{H_p^\gamma}^2 ds \right] < \infty.$$

We define the Banach spaces $\mathbb{L}_p(\tau)$, $\mathbb{L}_p(\tau, \ell_2)$ and $\mathcal{H}_p^\gamma(\tau)$ similarly. The following theorem plays the key role when we weaken condition (3.2) later in the next section.

Theorem 4.5 *Let $\tau \leq T$ be a stopping time. Fix a constant ε_1 so that $\varepsilon_1 = 0$ if $p = 2$, and $\varepsilon_1 > \alpha(1/2 - 1/p)$ if $p > 2$. Then, under Assumption 2.10 and (3.2), for any $f \in \mathbb{H}_p^\gamma(\tau)$, $h \in \mathbb{H}_p^{\gamma+\alpha/2}(\tau, \ell_2)$, $g^{\cdot, j} \in \mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(\tau, \ell_2)$, $1 \leq j \leq m$ and $u_0 \in U_p^{\gamma+\alpha/2-\alpha/p}$, equation (3.4) has a unique solution u in $\mathcal{H}_p^{\gamma+\alpha}(\tau)$, and for this solution*

$$\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(\tau)} \leq c \left(\|f\|_{\mathbb{H}_p^\gamma(\tau)} + \|h\|_{\mathbb{H}_p^{\gamma+\alpha/2}(\tau, \ell_2)} + \sum_{j=1}^m \|g^{\cdot, j}\|_{\mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(\tau, \ell_2)} + \|u_0\|_{U_p^{\gamma+\alpha-\alpha/p}} \right), \quad (4.27)$$

where $c = c(p, T, \delta)$ independent of τ .

Proof. First we prove the existence and (4.27). Obviously we have

$$\bar{f} := \mathbf{1}_{[0, \tau]} f \in \mathbb{H}_p^\gamma(T), \quad \bar{h} := \mathbf{1}_{[0, \tau]} h \in \mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2), \quad \bar{g}^{\cdot, j} := \mathbf{1}_{[0, \tau]} g^{\cdot, j} \in \mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(T, \ell_2).$$

Let $u \in \mathcal{H}_p^{\gamma+\alpha}(T)$ be the solution of (2.14) with \bar{f}, \bar{h} and \bar{g} instead of f, h and g respectively. Then, since $\tau \leq T$, we have $\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(\tau)} \leq \|u\|_{\mathcal{H}_p^{\gamma+\alpha}(T)}$, and by Theorem 3.6,

$$\begin{aligned} \|u\|_{\mathcal{H}_p^{\gamma+\alpha}(\tau)} &\leq c \left(\|\bar{f}\|_{\mathbb{H}_p^\gamma(T)} + \|\bar{h}\|_{\mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)} + \sum_{j=1}^m \|\bar{g}^{\cdot j}\|_{\mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(T, \ell_2)} + \|u_0\|_{U_p^{\gamma+\alpha/2-\alpha/p}} \right) \\ &= c \left(\|f\|_{\mathbb{H}_p^\gamma(\tau)} + \|h\|_{\mathbb{H}_p^{\gamma+\alpha/2}(\tau, \ell_2)} + \sum_{j=1}^m \|g^{\cdot j}\|_{\mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(\tau, \ell_2)} + \|u_0\|_{U_p^{\gamma+\alpha/2-\alpha/p}} \right). \end{aligned}$$

Now we prove the uniqueness. Let $u \in \mathcal{H}_p^{\gamma+\alpha}(\tau)$ be a solution of equation (2.14). Then obviously, $\mathbf{1}_{[0, \tau]} \cdot (\mathbb{D}u - a(\omega, t)\Delta^{\alpha/2}u) \in \mathbb{H}_p^\gamma(T)$, $\mathbf{1}_{[0, \tau]} \cdot \mathbb{S}_c u \in \mathbb{H}_p^{\gamma+\alpha/2}(T, \ell_2)$, $\mathbf{1}_{[0, \tau]} \cdot \mathbb{S}_d^{\cdot j} u \in \mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(T, \ell_2)$.

According to Theorem 3.6 we can define $v \in \mathcal{H}_p^{\gamma+\alpha}(T)$ as the solution of

$$\begin{aligned} dv &= (a(\omega, t)\Delta^{\alpha/2}v + \mathbf{1}_{[0, \tau]} (\mathbb{D}u - a(\omega, t)\Delta^{\alpha/2}u))dt + \sum_{k=1}^{\infty} \mathbf{1}_{[0, \tau]} \mathbb{S}_c^k u dW_t^k \\ &\quad + \sum_{k=1}^{\infty} \sum_{j=1}^m \mathbf{1}_{[0, \tau]} \mathbb{S}_d^{k, j} u dZ_t^k, \quad v(0) = u(0). \end{aligned} \quad (4.28)$$

Then for $t \leq \tau$, $d(u - v) = \Delta^{\alpha/2}(u - v)dt$ and $(u - v)(0) = 0$. Therefore by Theorem 2.9, we conclude that $u(t) = v(t)$ for all $t \leq \tau$ a.s.. By replacing u by v for $t \leq \tau$, from (4.28) we find that v satisfies

$$dv = \left(a\Delta^{\alpha/2}v + f\mathbf{1}_{[0, \tau]} \right) dt + \sum_{k=1}^{\infty} \mathbf{1}_{[0, \tau]} h^k dW_t^k + \sum_{k=1}^{\infty} \sum_{j=1}^m \mathbf{1}_{[0, \tau]} g^{k, j} dZ_t^k, \quad v(0) = u_0. \quad (4.29)$$

We proved that if $u \in \mathcal{H}_p^{\gamma+\alpha}(\tau)$ is a solution of equation (2.14) then $u(t) = v(t)$ for all $t \leq \tau$ a.s.. This proves the uniqueness of solution of equation (2.14) in the class $\mathcal{H}_p^{\gamma+\alpha}(\tau)$ because by Theorem 3.6 $v \in \mathcal{H}_p^{\gamma+\alpha}(T)$ is the unique solution of equation (4.29). The theorem is proved. \square

For a stopping time $\tau \leq T$ and $\gamma \in \mathbb{R}$, write $u \in \mathbb{H}_{p, \text{loc}}^\gamma(\tau)$ if there exists a sequence of stopping times $\tau_n \uparrow \infty$ so that $u \in \mathbb{H}_p^\gamma(\tau \wedge \tau_n)$ for each n .

The following is a weakened version of (3.2).

Assumption 4.6 *There exists an integer $N_0 \geq 1$ so that $\hat{c}_k < \infty$ for all integer $k > N_0$.*

Definition 4.7 *Let $u_0 \in U_p^{\gamma+\alpha-\alpha/p}$, $f(0) \in \mathbb{H}_p^\gamma(\tau)$, $h(0) \in \mathbb{H}_p^{\gamma+\alpha/2}(\tau, \ell_2)$ and $g^{\cdot j}(0) \in \mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(\tau, \ell_2)$, $1 \leq j \leq m$. We say that $u \in \mathcal{H}_{p, \text{loc}}^{\gamma+\alpha}(\tau)$ is a path-wise solution to (3.4) if the followings hold;*

(i) $u \in \mathbb{H}_{p, \text{loc}}^{\gamma+\alpha}(\tau)$ and $u(t)$ is right continuous with left limits in H_p^γ for $t < \tau$ (a.s.),

(ii) for any $\phi \in C_0^\infty(\mathbb{R}^d)$, the equality

$$\begin{aligned} (u(t, \cdot), \phi) = & (u_0, \phi) + \int_0^t a(\omega, s)(u(s, \cdot), \Delta^{\alpha/2} \phi) ds + \int_0^t (f(s, \cdot), \phi) ds \\ & + \sum_{k=1}^{\infty} \int_0^t (h^k(s, \cdot), \phi) dW_s^k + \sum_{k=1}^{\infty} \sum_{j=1}^m \int_0^t (g^{k,j}(s, \cdot), \phi) dY_s^{k,j} \end{aligned} \quad (4.30)$$

holds for all $t < \tau$ a.s..

Theorem 4.8 *Let $\tau \leq T$. Suppose that Assumptions 2.10 and 4.6 hold. Then for any $u_0 \in U_p^{\gamma+\alpha-\alpha/p}$, $f \in \mathbb{H}_p^\gamma(\tau)$, $h \in \mathbb{H}_p^{\gamma+\alpha/2}(\tau, \ell_2)$, $g^{\cdot,j} \in \mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(\tau, \ell_2)$, $1 \leq j \leq m$, there exists a unique path-wise solution $u \in \mathcal{H}_{p,\text{loc}}^{\gamma+\alpha}(\tau)$ to (3.4). In particular, if $\gamma + \alpha > d/p$, then the unique path-wise solution u is $C^{\gamma+\alpha-d/p}$ -valued process (for $t \leq \tau$) a.s..*

Proof. Step 1. First, additionally assume that (3.2) holds. Then the existence of path-wise solution under (3.2) in $\mathcal{H}_p^{\gamma+\alpha}(\tau)$ (hence in $\mathcal{H}_{p,\text{loc}}^{\gamma+\alpha}(\tau)$) follows from Theorem 4.5. Now we show that the pathwise solution is unique in $\mathcal{H}_{p,\text{loc}}^{\gamma+\alpha}(\tau)$. Let $u \in \mathcal{H}_{p,\text{loc}}^{\gamma+\alpha}(\tau)$ be a path-wise solution. Define $\tau_n = \tau \wedge \inf\{t : \int_0^t \|u\|_{H_p^{\gamma+\alpha}}^2 ds > n\}$. Then $u \in \mathbb{H}_p^{\gamma+\alpha}(\tau_n)$ and $\tau_n \uparrow \tau$ since $\int_0^t \|u\|_{H_p^{\gamma+\alpha}}^2 ds < \infty$ for all $t < \tau$, a.s. By Theorem 4.5,

$$\|u\|_{\mathcal{H}_p^{\gamma+\alpha}(\tau_n)} \leq c(T, d, \alpha) \left(\|f\|_{\mathbb{H}_p^\gamma(\tau_n)} + \|h\|_{\mathbb{H}_p^{\gamma+\alpha/2}(\tau_n, \ell_2)} + \sum_{j=1}^m \|g^{\cdot,j}\|_{\mathbb{H}_p^{\gamma+\alpha/2+\varepsilon_1}(\tau_n, \ell_2)} + \|u_0\|_{U_p^{\gamma+\alpha-\alpha/p}} \right).$$

By letting $n \rightarrow \infty$ we find that $u \in \mathcal{H}_p^{\gamma+\alpha}(\tau)$, and the uniqueness of the pathwise solution under (3.2) follows from the uniqueness result of Theorem 4.5.

Step 2. For the general case, note that for each $n > 0$ and $k \leq N_0$,

$$\widehat{c}_{k,n} := \left(\int_{\{z \in \mathbb{R}^m : |z| \leq n\}} |z|^2 \nu_k(dz) \right)^{1/2} \vee \left(\int_{\{z \in \mathbb{R}^m : |z| \leq n\}} |z|^p \nu_k(dz) \right)^{1/p} < \infty.$$

Consider Lévy processes $(Z_n^1, \dots, Z_n^{N_0}, Z^{N_0+1}, \dots)$ in place of (Z^1, Z^2, \dots) , where $Z_n^k (k \leq N_0)$ is obtained from Z^k by removing all the jumps that has absolute size strictly large than n . Note that condition (3.2) is valid with \widehat{c}_k replaced by $\widehat{c}_{k,n}$. By **Step 1**, there is a unique path-wise solution $v_n \in \mathcal{H}_p^{\gamma+\alpha}(\tau)$ with Z_n^k in place of Z^k for $k = 1, 2, \dots, N_0$. Let T_n be the first time that one of the Lévy processes $\{Z^k, 1 \leq k \leq N_0\}$ has a jump of (absolute) size in (n, ∞) . Define $u(t) = v_n(t)$ for $t < T_n \wedge \tau$. Note that for $n < m$, by **Step 1**, we have $v_n(t) = v_m(t)$ for $t < T_n \wedge \tau$. This is because, for $t < T_n \wedge \tau$, both v_n and v_m satisfy (4.30) with each term inside the stochastic integral multiplied by $1_{s < T_n}$ (and with Z_n^k , $k \leq N_0$, in place of Z^k). Thus u is well defined. By letting $n \rightarrow \infty$, one constructs a unique pathwise solution u in $\mathcal{H}_{p,\text{loc}}^{\gamma+\alpha}(\tau)$. The last claim follows from Sobolev embedding theorem. The theorem is proved. \square

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